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RECORDS OF PROCEEDINGS AT MEETINGS.

SESSION NOVEMBER, 1916–JUNE, 1917.

Thursday, November 2nd, 1916.

Annual General Meeting.

Prof. Sir J. LARMOR, President, in the Chair.

Present twenty-eight members and two visitors.

Prof. M. Kuniyeda was elected a member.

The Treasurer presented his Report. On the motion of Prof. A. S. Eddington, seconded by Mr. G. N. Watson, the Report was received.

Lt.-Col. A. Cunningham was appointed Auditor.

The Council and Officers for the ensuing Session were elected. The list is as follows:—President, Prof. H. M. Macdonald; Vice-Presidents, Mr. G. H. Hardy, Sir J. Larmor, and Major P. A. MacMahon; Treasurer, Dr. A. E. Western; Secretaries, Dr. T. J. I'A. Bromwich and Mr. G. T. Bennett; other members of the Council, Prof. W. Burnside, Mr. J. E. Campbell, Dr. S. Chapman, Mr. A. L. Dixon, Prof. H. Hilton, Prof. E. W. Hobson, Mr. A. E. Jolliffe, Prof. A. E. H. Love, Prof. J. W. Nicholson.

Sir J. Larmor spoke briefly on the early history of the Society, calling attention to its fiftieth anniversary, which had passed by without comment; he referred also to the losses of the Society during his term of office as President, and to the renewal of the associations of the Society with Burlington House.*

Prof. H. M. Macdonald then took the chair, and Sir J. Larmor delivered an address on "The Fourier Harmonic Analysis: its Practical

* The first meeting of the Society was held on January 16th, 1865, at University College. The following year, the Annual General Meeting was held (on November 8th, 1866) in the rooms of the Chemical Society at Burlington House: and the Society continued to meet in those rooms for four years.

Scope with Optical Illustration."* It was agreed that the address should be printed in the *Proceedings*.

The following papers were communicated, by title, from the Chair:—

*On Multiple Integration by Parts and the Second Theorem of the Mean: Prof. W. H. Young.

Moving Axes and their Uses in the Differential Geometry of Euclidean Space: Mr. E. H. Neville.

Areas and Conformal Representation: Mr. J. Hodgkinson.

ABSTRACTS.

The Fourier Harmonic Analysis: its Practical Scope with Optical Illustration

Sir JOSEPH LARMOR.

It is only under special circumstances that a function given by an unlimited graph is expressible by systems of unlimited trains of simple waves, with periods arithmetically discrete: the function must either be periodic itself, or compounded of periodic constituents, while sharper or more numerous irregularities in the graph involve constituent trains of higher and higher frequency. The Fourier integral, interpretable as representing systems of simple wave-trains having a continuous range of frequencies, is applicable only to the case of a limited graph, and under like reservations. If only a part of this graph is available for analysis, the ultimate element in the result is not an unending simple wave-train, but is a small though finite range of the integral, whose extent specifies the degree of precision or "purity" attainable in this partial analysis. The element represents what we may designate as a Fourier group of simple waves. Such a group is of limited extent, fading away towards front and rear; it has the important property of being unaltered in type by transmission through a dispersive medium. If we are concerned with an unlimited graph, such as a magnetic or meteorological record, all that we can effect in this connexion is to analyse, as specimens, different finite reaches of it into their Fourier integrals: if a period or short range of periods comes out conspicuous in them all, a presumption is afforded that it is a

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* Printed in this volume.

permanent feature in the material that is analysed, but it may represent, however, only broken periodicity after the manner of natural radiation (*infra*). But any period that is conspicuous in the analysis of only one region of the graph is presumably either accidental or due to local and transient causes. If the analyses of the different sample reaches give different results, the inference would be that the graph is wholly fortuitous, with no permanent features at all. Sound views on this matter are of fundamental importance for observational subjects such as meteorology: the question has hardly yet been faced, whether it is in any given case essential or helpful to go on piling up numerical details of observations, with careful avoidance of any breach in the records, in the hope that posterity may be able to make some use of them.

A different type of problem is the harmonic analysis of material possessing partial or broken periodicities, typified by the case of natural radiation, which consists of a vast complex of transient trains of waves to some extent periodic, emitted from time to time by the crowd of separate molecules of the source. Here both the natures of the partial trains, and the statistics of their fortuitous distribution along the beam of light, are involved: and a clear understanding as regards both these matters is necessary to the exact study of the constitution and transformations of natural radiation.

On Multiple Integration by Parts and the Second Theorem of the Mean

Prof. W. H. YOUNG.

In this paper the general Theorem of Integration by Parts is proved in a form which takes account of recent progress in the theory of integration, and the Second Theorem of the Mean, in an extended form, and available for any number of variables, is deduced from it. The method employed involves the theory of integration with respect to a function of bounded variation, and depends for its generality on a property of these functions recently brought to light, which, in the case of two variables, is that the discontinuities of the function lie on a countable set of lines parallel to the two axes. An extension to the case of non-absolutely convergent integrals is also briefly indicated. Two main forms of the Formula of Integration are obtained. Taking for simplicity of statement the case of two variables, and supposing $f(x, y)$ and $g(x, y)$ both to have bounded

variation, and one of them to be continuous, the first of these two forms is the following :—

$$\int_{0,0}^{a,b} g(x,y)\,df(x,y) = [fg]_{0,0}^{a,b} - \int_{x=0}^{a} [f\,dg]_{y=0}^{b} - \int_{y=0}^{b} [f\,dg]_{x=0}^{a} + \int_{0,0}^{a,b} f\,dg,$$

where, with the usual notation, $[f]_{y=0}^{b}$ means the increment of $f(x, y)$ with respect to y, viz., $f(x, b) - f(x, 0)$, and the same notation is used for the higher increments, so that

$$[f]_{0,0}^{a,b} = f(0,0) - f(a,0) - f(0,b) + f(a,b).$$

The second form is deduced from the first by an algebraic transformation, and has as many variants as the fundamental hyper-rectangle has corners. That in which the value of g at the origin explicitly occurs is for two variables the following :—

$$\int_{0,0}^{a,b} g(x,y)\,df(x,y)$$

$$= g(0,0)[f]_{0,0}^{a,b} + \int_{0,0}^{a,b} [f]_{x,y}^{a,b}\,dg(x,y) + \int_{0}^{a} [f]_{x,0}^{a,b}\,dg(x,0) + \int_{0}^{b} [f]_{0,y}^{a,b}\,dg(0,y)$$

By supposing one or both of the functions $f(x, y)$, $g(x, y)$, to be integrals with respect to xy, or more generally with respect to any function of bounded variation, special, but very important, cases of these formulæ are obtained, in virtue of a Theorem of Substitution proved by the author in a previous paper. The Theorem of the Mean is deduced from the second form of the Formula for Integration by Parts, and is given both in Weierstrass's and in Bonnet's form. In the latter form it states that

$$\int_0^a g\,df = g(a)\int_X^a df$$

[where the single letter or figure stands for the corresponding ensemble, *e.g.*, a for $(a_1, a_2, \ldots, a_n)$], X being a certain point in the hyper-rectangle $(0, a)$.

If, on the other hand, g is not known to be positive, but has all its increments positive, we have the second form, as follows :—

$$\int_0^a g\,df = g(0)\int_0^a df + [g(a) - g(0)]\int_X^a df$$

Each of these two equations has as many variants as the two forms of the Formula for Integration by Parts, 2^n in number. Other variants can be obtained, for monotone functions of bounded variation not included in the

2^n types to which these variants apply. These are less elegant in form, and less useful in practice, since, as is shown in the paper, all the general types of such functions can be expressed as the algebraic sum of a number of functions all belonging to the special types.

The paper concludes with a modification in the direction of greater simplicity of a form of the Theorem of the Mean obtained by Lebesgue. The formula is applicable when g is any function of bounded variation, and runs as follows:—

$$\left| \int_0^a f(x)\, g(x)\, dx - g(0) \int^a f dx \right| \leqslant M V(a),$$

where V is the total variation of the function g, and M is the upper bound of $\int_x^a f dx$ in the hyper-rectangle $(0, a)$.

The General Theory of Moving Axes in Euclidean Space of Three Dimensions

Mr. ERIC H. NEVILLE.

The theory developed in this paper is general in that the angles of the moving frame are not restricted to be constant. The motion of a frame dependent on any finite number of variables is analysed, the effects of the motion on the shape of the frame and on its use as a frame of reference are determined, existence theorems are proved, and the construction of a frame with a given motion is discussed.

The primitive theorems, of which an announcement was made to the Fifth International Congress of Mathematicians, and a proof is shortly to appear in the *Quarterly Journal of Mathematics*, are as follows:—

The motion of a frame $OABC$ dependent on a variable t can be expressed by means of three translations, which are the components relative to $OABC$ itself of the velocity of O with respect to t, and six spins ϕ_1, ψ_1, ψ_2, θ_2, θ_3, ϕ_3, of which ϕ_1 and ψ_1 are the components about OB and OC of any spin which OA can be supposed to have.

If a vector has with reference to $OABC$ the components x, y, z and the projections l, m, n, and if the vector and the frame alike depend on a variable t, then the rate of change of the vector has for its projections on the axes of the frame

$$\frac{dl}{dt} - \Upsilon(y\psi_1 - z\phi_1), \quad \frac{dm}{dt} - \Upsilon(z\theta_2 - x\psi_2), \quad \frac{dn}{dt} - \Upsilon(x\phi_3 - y\theta_3),$$

where Υ is the sine of the solid angle of the frame.

The simplicity of the second of these theorems both justifies the analysis made in the first and ensures the practicability of the resultant theory.

When there are several independent variables, translations and spins give place to translation coefficients and spin coefficients, which are interrelated; to find a complete set of equations connecting the coefficients is easy, but in proving that the set is complete we have to overcome the peculiar difficulty that the equations as they are naturally found and used involve explicitly the angles of the frame, and are therefore meaningless if the frame is not known to exist.

The theory has proved exceedingly fertile in the regions where it has been used.

Thursday, December 14*th*, 1916.

Prof. H. M. MACDONALD, President, in the Chair.

Present seventeen members and two visitors.

Mr. Rohani Thirunal was elected a member.

The Auditor (Lt.-Col. A. Cunningham) presented his Report. On the motion of Mr. G. N. Watson, seconded by Mr. L. J. Mordell, the Treasurer's Report was adopted, and the thanks of the Society were voted to the Treasurer and to the Auditor.

Messrs. G. H. Hardy and S. Ramanujan communicated two papers: (i) "Proof that almost all Numbers n are composed of about $\log\log n$ Prime Factors," (ii) "An Asymptotic Formula for the Number of Partitions of a Number."

Mr. G. N. Watson read a paper, "The Harmonic Functions associated with the Parabolic Cylinder (Second Paper)."

Mr. G. N. Watson also made an informal communication "On the Theory of Numbers."

The following papers, in the absence of their authors, were communicated, by title, from the chair:—

On the Efficiency of a Surface of Discontinuity regarded as a Propeller: Prof. W. Burnside.

Orbits Asymptotic to an Isosceles Triangle Solution of the Problem of Three Bodies: Prof. D. Buchanan.

Diffraction of Waves by a Wedge of any Angle: Prof. H. S. Carslaw.

*On Two Theorems of Combinatory Analysis and Some Allied Identities: Prof. L. J. Rogers.

*(i) On the Internal Structure of a Set of Points in Space of any Number of Dimensions, (ii) On the Inherently Crystalline Structure of a Function of any Number of Variables: Prof. W. H. Young and Mrs. Young.

ABSTRACTS.

Proof that almost all Numbers n are composed of about log log *n Prime Factors*

Messrs. G. H. HARDY and S. RAMANUJAN.

A number n is described in popular language as a *round* number if it is composed of a considerable number of comparatively small factors: thus $1200 = 2^4.3.5^2$ would generally be said to be a round number. It is a matter of common experience that round numbers are exceedingly rare. The fact may be verified by anybody who will make a practice of factorising numbers, such as the numbers of taxi-cabs or railway carriages, which are presented to his attention in moments of leisure. The object of this paper† is to provide the mathematical explanation of this phenomenon.

Let $\pi_\nu(x)$ denote the number of numbers which do not exceed x and are formed of exactly ν prime factors. There is an ambiguity in this definition, for we may count multiple factors multiply or not. But the results are substantially the same on either interpretation.

It has been proved by Landau‡ that

$$\pi_\nu(x) \sim \frac{x}{\log x} \frac{(\log\log x)^{\nu-1}}{\nu-1!} \tag{1}$$

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* Printed in this volume.

† The paper has been published in the *Quarterly Journal of Mathematics*, Vol. 48, pp. 76–92.

‡ See *Handbuch der Lehre von der Verteilung der Primzahlen*, pp. 203–213.

as $x \to \infty$, for every fixed value of ν. It is moreover obvious that

$$[x] = \pi_1(x) + \pi_2(x) + \pi_3(x) + \ldots, \tag{2}$$

and

$$x = \frac{x}{\log x}\left\{1 + \log\log x + \frac{(\log\log x)^2}{2!} + \ldots\right\}. \tag{3}$$

Landau's result shews that there is a certain correspondence between the terms of the series (2) and (3). The correspondence is far from exact. The first series is finite, for it is obvious that $\pi_\nu(x) = 0$ if $\nu > (\log x)/(\log 2)$; and the second is infinite. But it is reasonable to anticipate a correspondence accurate enough to throw considerable light on the distribution of the numbers less than x in respect of the number of their prime factors.

The greatest term of the series (3) occurs when ν is about $\log\log x$. And if we consider the block of terms for which

$$\log\log x - \phi(x)\sqrt{(\log\log x)} < \nu < \log\log x + \phi(x)\sqrt{(\log\log x)}, \tag{4}$$

where $\phi(x)$ is any function of x which tends to infinity with x, we find without difficulty that it is these terms which contribute almost all the sum of the series: the ratio of their sum to that of the remaining terms tends to infinity with x.

In this paper we shew that the same conclusion holds for the series (2). Let us consider all numbers n which do not exceed x, and denote by x_P the number of them which possess a property $P(n, x)$: this property may be a function of both n and x, or of one variable only. If then $x_P/x \to 1$ when $x \to \infty$, we say that *almost all numbers less than x possess the property P*. And if P is a function of n only, we say simply that *almost all numbers possess the property P*. This being so, we prove the following theorems.

1. *Almost all numbers n less than x are formed of more than*

$$\log\log x - \phi(x)\sqrt{(\log\log x)}$$

and less than

$$\log\log x + \phi(x)\sqrt{(\log\log x)}$$

prime factors.

2. *Almost all numbers n are formed of more than*

$$\log\log n - \phi(n)\sqrt{(\log\log n)}$$

and less than

$$\log\log n + \phi(n)\sqrt{(\log\log n)}$$

prime factors.

In these theorems ϕ is any function of x (or n) which tends to infinity with its argument: and either theorem is true in whichever manner the factors of n are counted. The only serious difficulty in the proofs lies in replacing Landau's asymptotic relations (1) by inequalities valid for all values of ν and x.

Since $\log\log n$ tends to infinity with extreme slowness, the theorems are fully sufficient to explain the observations which suggested them.

The Harmonic Functions associated with the Parabolic Cylinder

(Second Paper)

Mr. G. N. WATSON.

The object of this paper is to investigate the asymptotic expansions of the parabolic cylinder functions $D_n(z)$, for large values of the order and all values of the argument, by means of the method of steepest descents. These functions, together with the Bessel functions, already investigated by Debye, form a complete set of functions for which the method is peculiarly well adapted. I also investigate the nature of the convergence of series of the type $\Sigma a_n D_n(2z\sqrt{n})$, which are analogous to Kapteyn's series of Bessel functions. In some respects the theorems contained in this paper require proofs of a distinctly more elaborate character than the corresponding theorems for Bessel functions.

On the Internal Structure of a Set of Points in Space of any Number of Dimensions

Prof. W. H. YOUNG and GRACE CHISHOLM YOUNG.

The present communication deals with that part of the Theory of Sets of Points in the Plane and Higher Space which corresponds to the classification of the limiting points of a set on the straight line into those which are limits on one side only, or on both sides (descriptive property), as well as into those which are, and those which are not, the end-points of intervals containing only a sub-set of the given set of content zero (metrical property). While on the straight line the matter is very simple,

in the plane and higher space it involves little less than a new chapter in the Theory of Sets of Points.

The matter is very fruitful of consequences in the Theory of Functions of Two or More Variables, some of which are developed in an accompanying memoir.

The new theory is built on certain preliminary theorems dealing with a set of points, each of which is the appropriate corner of a rectangle, having inside it no points of the set, and whose sides are parallel to fixed rectangular axes. Calling such a rectangular void a *flag*, it is shown that, *if at each point of the set there is such a flag* (*e.g.*, *one having the point in question for left-hand bottom corner always*), *then at each point there is also a second flag, namely in the opposite quadrant.* The sets to be considered fall accordingly into two types, those having at each point four flags, and those having but two.

We then have the following theorems :—

(1) *A set with four flags lies on a countable set of parallels to the axes.*

(2) *A set with two flags lies on a countable aggregate of non-oscillatory curves, and is therefore of content zero.*

In each case the sets are, of course, nowhere dense.

The flags here do not include any of their bounding lines. If one pair of these are included, the lines in (1) are parallel to the remaining pair. Hence if both pairs are included the set (1) is countable.

Using these results we are able to prove the following general theorem, in which the voids are no longer rectangular, but triangular with any angles :—

The points at which a plane set possesses triangular voids is not only nowhere dense, but lies on a countable aggregate of non-oscillatory curves, constituting accordingly a set of zero content.

Hence, still more generally, *a plane set which can be divided up into a countably infinite aggregate of sets, each possessing at each of its points a triangular void, is a set of the first category, of content zero, and lies on a countable aggregate of non-oscillatory curves.*

If S be a plane set, and C its complementary set, we may accordingly divide up the points of the plane into four classes :—

(1) *internal points of the set S ;*

(2) *internal points of the complementary set C ;*

(3) *boundary points of both S and C, at which either S or C has a*

triangular void: these lie on a countable aggregate of non-oscillatory curves;

(4) *stickle points of both S and C;* which are boundary points of both sets, having the property that in every triangle, with one of them as vertex, there are points both of S and of C.

In particular, if S is a region, its boundary points fall into two classes; firstly, those which, being the corners of triangles containing inside them only points of the region, form the *natural boundary* of the region; and, secondly, *stickle points* of the region. The natural boundary is of content zero, and lies on a countable set of non-oscillatory curves; the stickle points may form a set of positive content.

Passing to the metrical point of view, we then obtain the theorem that *any plane set may be divided up into a sub-set of the first category and content zero, lying on a countable aggregate of non-oscillatory curves, and a sub-set having the property that, inside every triangle with a point of this latter sub-set as corner, the points of this latter sub-set form a set of positive content.*

The generalisation of these theorems for space of three or more dimensions is immediate.

On the Inherently Crystalline Structure of a Function of any Number of Variables

Prof. W. H. YOUNG and GRACE CHISHOLM YOUNG.

In this memoir the new theorems communicated in the companion paper are applied to the investigation of the symmetrical relations which hold in general between the limits of a function at a point, and the discussion of the distribution of the points at which this symmetry is more or less imperfect.

The paper is a sequel to other papers by W. H. Young on the subject, and completes the provisional results for the plane and higher space already communicated by him to the Society. The methods employed are of the same nature as before, but are applied more thoroughly, and therefore with greater success.

On the straight line the results were strikingly simple; the points at which there was a distinction of right and left were countable At such

points the structure of the function may be said to be *amorphic*. At the remaining points the symmetry was perfect, all the limits of $f(x+h)$ being the same as those of $f(x-h)$ (the increment h being positive and approaching zero).

In the plane there are other varieties of structure besides amorphism and perfect symmetry. Moreover there is no one obvious definition of symmetry, as was the case on the straight line. We find that *there is no distinction of right and left, except on a countable aggregate of vertical straight lines.* This involves only the variation of x. If, however, we let both x and y vary (the increments h and k being positive and approaching zero), we have to compare the sets of limits of $f(x \pm h, y \pm k)$ in the different completely open quadrants formed by the parallels to the axes through the point. We find then that *there is perfect symmetry with regard to quadrants except at points forming a set of the first category and content zero, lying on a countable aggregate of non-oscillatory curves.* The last statement is quite new; the first had already been proved, and the second foretold by W. H. Young. *The points of amorphism with regard to quadrants form again a countable set.* There are, however, intermediate forms of imperfect symmetry, in which two or three, but not four of the sets of limits are identical; these are discussed in detail, and found to lie in various cases on a countable set of vertical or horizontal lines, or to be countable.

Similar results are given for octants, &c. Finally we have the following theorem :—

All the limits of $f(x, y)$ at a point P may be obtained by means of sequences tangential at P to a half-ray fixed in direction independently of P, unless P lies on a certain countable set of non-oscillatory curves, independent of the chosen direction.

The generalisation for functions of more variables is immediate.

Thursday, January 18th, 1917.

Prof. H. M. MACDONALD, President, in the Chair.

Present sixteen members.

Mr. E. Lindsay Ince was elected a member.

In the absence of Mr. Bennett, on war-work, Dr. Bromwich read the Secretary's Report :—During the Session 1915–1916, twelve members

were elected, eight died, and one resigned. At the date of the Annual General Meeting (November 2nd, 1916) the number of members was 302, against 299 at the beginning of the previous Session. The Royal Society of Sciences at Amsterdam (*Koninklije Akademie van Wetenschappen te Amsterdam*) had been added to the list of Societies with which this Society exchanges Proceedings.

Mr. G. H. Hardy read a paper (written in conjunction with Mr. S. Ramanujan), "Asymptotic Formulæ in Combinatory Analysis."

Prof. M. J. M. Hill read a paper, "On the Singular Solutions of Ordinary Differential Equations of the First Order with Transcendental Coefficients." *

In the absence of the authors, the following papers were read by title from the Chair:—

The Nature of a Moving Electric Charge and its Lines of Electric Force: Mr. H. Bateman.

On the Expansion of the Variables of a Hypergeometric Equation in Terms of the Ratio of Two Solutions: Prof. L. J. Rogers.

A Problem in the Theory of Diffraction: Prof. H. J. Priestley.

ABSTRACTS.

On the Singular Solutions of Ordinary Differential Equations of the First Order with Transcendental Coefficients

Prof. M. J. M. HILL.

The methods of deducing the singular solutions of ordinary differential equations of the first order, both from the complete primitive and from the differential equation itself, were first given by Lagrange in a memoir which is printed in the fourth volume of his works.

Commencing with the complete primitive of an equation in the form

$$y = f(x, a),$$

he showed that a solution could be found by eliminating a between

$$y = f(x, a) \quad \text{and} \quad 0 = \frac{df(x, a)}{da}.$$

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* Printed in this volume.

He then went on to say that it was easy to prove that no other possible combinations could exist which would supply integrals of this kind not contained in the complete integral. This is in effect an assumption of the converse of the proposition he had proved.

In like manner after showing that the singular solution of the differential equation

$$\phi(x, y, p) = 0,$$

must also satisfy $\dfrac{D\phi}{Dp} = 0$ and $\dfrac{D\phi}{Dx} + p\dfrac{D\phi}{Dy} = 0,$

he said that, though this result had been obtained on the hypothesis that ϕ contained no transcendental functions, it was not difficult to convince oneself that the same conclusion would hold whatever the nature and the form of ϕ might be. This again is an assumption of the converse proposition to that which he had proved.

The first of the two converse propositions assumed by Lagrange has, I believe, never been proved.

The object of this paper is to give examples showing that the second of those two propositions is not universally true.

Taking as a complete primitive the equation

$$f(x, y) + \phi(w)\,\psi(x, y) = c,$$

where w is some function of x, y, which is such that neither

$$\frac{\partial w}{\partial x} = 0 \quad \text{nor} \quad \frac{\partial w}{\partial y} = 0$$

is a consequence of $w = 0$; where $\phi(w)$ vanishes when w vanishes; where $\phi'(w)$ is infinite when w vanishes; where $f(x, y)$ neither vanishes, nor reduces to a constant, nor becomes infinite when w vanishes; and where $\psi(x, y)$ neither vanishes nor becomes infinite when w vanishes; then the differential equation is

$$\frac{1}{\phi'(w)}\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}p\right) + \frac{\phi(w)}{\phi'(w)}\left(\frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y}p\right) + \left(\frac{\partial w}{\partial x} + \frac{\partial w}{\partial y}p\right)\psi(x, y) = 0.$$

It is clear that $w = 0$ is an integral of this differential equation, and that it cannot be obtained from the complete primitive by giving to c any constant value, so that it is a singular solution.

The condition $\dfrac{D\phi}{Dp} = 0$ is, in this case,

$$\frac{1}{\phi'(w)}\frac{\partial f}{\partial y} + \frac{\phi(w)}{\phi'(w)}\frac{\partial \psi}{\partial y} + \frac{\partial w}{\partial y}\psi(x, y) = 0,$$

which, when $w = 0$, becomes

$$\frac{\partial w}{\partial y}\psi(x, y) = 0,$$

and this cannot be satisfied, since, when $w = 0$, neither $\partial w/\partial y$ nor $\psi(x, y)$ vanish.

The most important cases are those in which

$$\phi(w) = w^n, \quad 0 < n < 1;$$

$$\phi(w) = (\log w)^{-n}, \quad n > 0;$$

and

$$\phi(w) = w(\log w)^n, \quad n > 0.$$

The simplest examples studied in the paper are

$$p = y(\log y)^2 \quad \text{and} \quad p = (1+\log y)^{-1}:$$

$y = 0$ is a singular integral of each, and it is impossible to satisfy the condition

$$\frac{D\phi}{Dp} = 0.$$

The Nature of a Moving Electric Charge and its Lines of Electric Force

Mr. H. BATEMAN.

A type of vector field is considered in which a moving singularity can be regarded as the source of all the electricity in the field. The field can be chosen so that it reduces to an ordinary electrostatic field when the singularity becomes stationary; it includes Liénard's electromagnetic field of a moving point charge as a particular case. The lines of electric force in the field can be obtained by solving a Riccatian equation. The motion of the lines of force is described with the aid of a succession of infinitesimal transformations, and it is found that the same succession of infinitesimal transformations can be associated with a number of moving point charges, whose motions are co-ordinated in a certain way. It is also found that some of the lines of electric force from these different point charges are permanently incident with one another, and that when the point charges are of Liénard's type, the co-ordinated motion may be represented

by means of the orthogonal trajectories of a family of hyperspheres in a space of four dimensions.

The fields of Thomson's type (which can be described by means of moving Faraday tubes) are discussed, and are shown to be derivable from families of mutually orthogonal systems of surfaces in a space of four dimensions.

A Problem in the Theory of Diffraction

Prof. H. J. PRIESTLEY.

The object of the paper is to determine a distribution of sources over a circular area on a plane screen that will make the potential constant over the area.

It is shewn that a distribution $-\sigma e^{-\iota nt}$ over a circular strip of radius a, and breadth δa, gives a potential ϕ at a distance r from the centre, and angular distance θ from the axis, where

$$\phi = -\iota\sigma\delta a \,.\, e^{-\iota nt}\, \pi^{\frac{1}{2}} (a/r)^{\frac{1}{2}} \sum_{p=0}^{\infty} C_p J_{2p+\frac{1}{2}}(ka) H_1^{2p+\frac{1}{2}}(kr) P_{2p}(\mu) \quad (r > a),$$

or

$$-\iota\sigma\delta a \,.\, e^{-\iota nt}\, \pi^{\frac{1}{2}} (a/r)^{\frac{1}{2}} \sum_{p=0}^{\infty} C_p J_{2p+\frac{1}{2}}(kr) H_1^{2p+\frac{1}{2}}(ka) P_{2p}(\mu) \quad (r < a),$$

where

$$\mu = \cos\theta,$$

$$C_p = (-1)^p (2p+\tfrac{1}{2})\, \Gamma(p+\tfrac{1}{2})/p!,$$

and $H_1^{2p+\frac{1}{2}}(ka)$ is Hankel's first cylinder function.

With the aid of these results it is shewn that the potential at a point of a circular disc of radius a, due to a distribution $-\psi(r)\, e^{-\iota nt}$ is given by

$$r^{\frac{1}{2}}\phi \,.\, e^{\iota nt} = \iota \sum_{0}^{\infty} a_p J_{2p+\frac{1}{2}}(kr) \left[\int H_1^{2p+\frac{1}{2}}(kr)\, r^{\frac{1}{2}}\, \psi(r)\, dr \right]_{r=a},$$

where

$$a_p = (2p+\tfrac{1}{2}) \left[\Gamma(p+\tfrac{1}{2})/p! \right]^2,$$

and the value of the indefinite integral used is that which contains no constant term when expanded in powers of r.

To make ϕ constant it is necessary to choose $\psi(r)$, so that

$$\left[\int H_1^{2p+\frac{1}{2}}(kr)\, r^{\frac{1}{2}}\, \psi(r)\, dr \right]_{r=a} = p!/\Gamma(p+\tfrac{1}{2}).$$

A formal expression for $\psi(r)$ is given as the quotient of two infinite determinants.

Thursday, February 1st, 1917.

Prof. H. M. MACDONALD, President, in the Chair.

Present sixteen members and two visitors.

Major MacMahon read two papers, *(i) "The Significance of a certain Algebraic Fraction in the Theory of Distributions," (ii) "The Number of Ways of Pairing Off the Members of Two Identical Sets of Different Quantities."

Mr. W. H. Salmon read a paper, "Curves of Constant Torsion," and exhibited models of two such curves, one being a closed rational algebraic curve.

Informal communications were made by Messrs. A. Cunningham, E. H. Neville, and L. J. Mordell.

ABSTRACT.

On Curves of Constant Torsion

Mr. W. H. SALMON.

This paper obtains (i) an infinite series of periodic curves of constant torsion traced on rational algebraic cylinders, the circular helix being the lowest of the series; (ii) an infinite series of rational algebraic curves of constant torsion, the second series being a particular case of the first when the periodic length is zero.

The equations

$$\left.\begin{aligned} x &= \frac{1}{\tau}\int \frac{BC'-B'C}{A^2+B^2+C^2}\,dt, \\ y &= \frac{1}{\tau}\int \frac{CA'-C'A}{A^2+B^2+C^2}\,dt, \\ z &= \frac{1}{\tau}\int \frac{AB'-A'B}{A^2+B^2+C^2}\,dt, \end{aligned}\right\}$$

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* Printed in this volume.

are known to represent a curve of constant torsion τ for which A, B, C are proportional to the direction-cosines of the binormal.

One particular type of such curves is given by writing

$$\left.\begin{aligned} A &= a\cos(bt+c) \\ B &= a\sin(bt+c)\cos(\beta t+\gamma), \\ C &= a\sin(bt+c)\sin(\beta t+\gamma). \end{aligned}\right\}$$

If $b = 0$, we obtain the circular helix. If b is finite, we may without loss of generality make $a = 1$, $b = 1$, $c = 0$. Considering the case when $\gamma = 0$, and β is any integer except ± 2, we have the series (i).

This series may also be regarded as a particular case of the series of curves of constant torsion given by

$$\left.\begin{aligned} A &= \Sigma_{r=1}^{r=2n+1} a_r \cos \alpha_r t, \\ B &= \Sigma_{r=}^{r=2n+1} a_r \sin \alpha_r t, \\ C &= \Sigma_{r=0}^{r=n} b_r \cos r\delta t, \end{aligned}\right\}$$

where $\alpha_1, \alpha_2, \ldots, \alpha_{2n+1}$ are an odd number of integers in arithmetical progression of common difference δ unequal to either $\pm\alpha_1, \pm\alpha_2, \ldots$, or $\pm\alpha_{2n+1}$, and where the values of the coefficients $a_1, a_2, \ldots, a_{2n+1}, b_0, b_1, \ldots, b_n$, are restricted by $2n$ equations. This series of periodic curves of constant torsion is reduced to the series (ii) by imposing another restriction upon the coefficients.

The above curves may be shown to include as particular cases some of the curves of constant torsion already discovered by M. Lyon and M. Fabry.

Thursday, March 1st, 1917.

Mr. G. H. HARDY, Vice-President, in the Chair.

Present also fourteen members.

Mr. A. E. Jolliffe read a paper, "Some Properties of a Quadrangle formed by the Points of Contact of the Tangents drawn to a Nodal Cubic from any Point."

Mr. E. H. Neville made an informal communication, "A Circular Indicatrix of the Curvature of a Surface."

ABSTRACTS.

*Asymptotic Formulæ in Combinatory Analysis**

Mr. G. H. HARDY and Mr. S. RAMANUJAN.

A preliminary account of some of the contents of this paper appeared in the *Comptes Rendus* of January 2nd, 1917. The paper contains a full discussion and proof of the results there stated. The asymptotic formula for $p(n)$, the number of unrestricted partitions of n, of which only the first three terms were given, is completed; and it is shown that, by taking a number of terms of order $\sqrt{n}$, the *exact* value of $p(n)$ can be obtained for all sufficiently large values of n. Some account is also given of actual or possible applications of the method used to other problems in Combinatory Analysis or the Analytic Theory of Numbers.

A Circular Indicatrix of the Curvature of a Surface

Mr. ERIC H. NEVILLE.

Whatever method is adopted for their calculation, the normal curvature κ_n and the geodesic torsion ς_g of a surface along a tangent OT through a point O take the forms

$$\kappa_n = D - U\cos 2\xi + V\sin 2\xi, \qquad \varsigma_g = U\sin 2\xi + V\cos 2\xi,$$

where ξ is an angle from a fixed tangent to OT and D, U, V are dependent on the position of O and on the reference tangent; the expressions found for D, U, V depend of course on the method that is being used, but that κ_n and ς_g are given by a pair of formulæ of the type stated is always apparent at an early stage. Writing

$$U = E\cos 2\eta, \qquad V = E\sin 2\eta, \qquad \zeta = \xi + \eta,$$

where E is not negative, we have

$$\kappa_n = D - E\cos 2\zeta, \qquad \varsigma_g = E\sin 2\zeta,$$

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* See *Records of Proceedings*, December 14th, 1916, and January 18th, 1917.

where D, E depend only on the position of O, and ξ is an angle to OT from a tangent that is determinate unless E is zero.

A sketch of one proof of formulæ for κ_n and s_g is essential to shew how little preliminary algebra we assume in starting from those given above. Suppose the surface referred to fixed rectangular axes to have the equation

$$\phi(x, y, z) = 0;$$

the normal at the current point O has cosines $G\phi_x$, $G\phi_y$, $G\phi_z$, where

$$G^{-2} = \phi_x^2 + \phi_y^2 + \phi_z^2,$$

and if a unit sphere is described about the origin of coordinates, the Gaussian image of O has these cosines for its coordinates, and therefore as O moves with velocity of components x', y', z', the image has velocity of components

$$G(\phi_{xx}x' + \phi_{xy}y' + \phi_{xz}z') + G'\phi_x, \quad G(\phi_{yx}x' + \phi_{yy}y' + \phi_{yz}z') + G'\phi_y, \quad G(\phi_{zx}x' + \phi_{zy}y' + \phi_{zz}z') + G'\phi_z.$$

If O is moving with unit speed in a direction OT, the projection of the velocity of the image* in the direction reverse to OT is κ_n, and the projection of the same velocity in the direction with which OT makes a positive right angle is s_g. If we suppose OT to have cosines l, m, n, and any other direction OS on the surface to have cosines p, q, r, then since

$$\phi_x p + \phi_y q + \phi_z r = 0,$$

the projection in the direction OS of the velocity of the image corresponding to motion of O with unit speed in the direction OT is

$$G(\phi_{xx}, \phi_{yy}, \phi_{zz}, \phi_{yz}, \phi_{zx}, \phi_{xy} \Big| l, m, n \Big| p, q, r),$$

and because this function is symmetrical and bilinear, its value in terms of angles ξ, χ which OT, OS make with a fixed direction on the surface is of the form

$$-P\cos\xi\cos\chi - V(\cos\xi\sin\chi + \sin\xi\cos\chi) - Q\sin\xi\sin\chi,$$

that is, of the form

$$-D\cos(\xi-\chi) + U\cos(\xi+\chi) - V\sin(\xi+\chi),$$

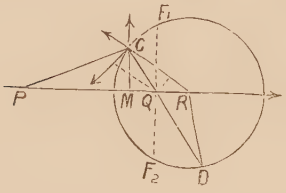
whence taking χ in turn as $\xi-\pi$ and $\xi-\frac{1}{2}\pi$, we have κ_n and s_g given in the forms asserted.

On a directed line through a fixed point P take the point R such that to an assigned scale PR is equal to D, with R as centre and radius E describe a circle, and let C be the point on the circle such that the angle from RP to RC in the negative direction is 2ξ. Then taking the point C to correspond to the tangent OT, *the normal curvature and the geodesic torsion along OT are measured by PM and MC, where M is the foot of the perpendicular from C on PR.* The simple construction given here

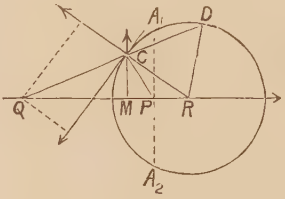
* If OE' is the direction (on the surface) with which OT makes a positive right angle, κ_n, s_g are the components about OE', OT of the spin of the tangent plane; the velocity of the image has these components in the directions with which OE', OT make positive right angles, that is, in the direction reverse to OT and the direction OE'. The actual torsion s of a curve on the surface is the component about OT of the spin of the osculating plane of the curve, and therefore s_g exceeds s by the rate of change of an angle from the osculating plane to the tangent plane, and this is equal to the rate of change of an angle from the principal normal to the normal to the surface.

exhibits all the elementary properties of κ_n and ς_g, and the figure contains representations of other magnitudes of importance; in particular, because PM and MC are at right angles, the length PC measures the spin of the surface along OT, or, in other words, the speed of the Gaussian image of O when O moves with unit speed along a curve touching OT, and the angles between CM and CP are the conjugate angles of OT, that is, the angles between OT and the tangent conjugate to OT.

The tangent at right angles to OT is represented by the point diametrically opposite to OT, and D is therefore seen to be half the sum of the normal curvatures along any two perpendicular tangents; the product K of the principal curvatures is the power of P with respect to the circle,



K positive.



K negative.

and E appears as half the difference between the principal curvatures. If K is negative, there are distinct real tangents along which the normal curvature is zero, namely, the tangents represented by the points A_1, A_2 in which the ordinate through P meets the circle; these tangents give equal and opposite values to ς_g, the square of the values being $-K$, whence a theorem of Enneper's. If K is positive, there are no real asymptotic tangents, but the points of contact F_1, F_2 of the tangents from P to the circle represent the tangents known as the characteristic tangents; these may be defined as the tangents for which the conjugate angle has its greatest value, and it is evident from elementary properties of the circle that they are the tangents along which κ_n has the value K/D.

Since RPC is complementary to a conjugate angle of OT, we can see without any examination of the signs of angles that the tangent conjugate to OT is represented either by the second point in which PC meets the circle, or by the reflection of this point or of C itself in PR; in the particular case in which P is inside the circle and C is at A_1, both the reflection of C in PR and the second point in which PC meets the circle are at A_2, and therefore in general the conjugate of OT is represented by the reflection in PR of the second point in which PC meets the circle; that is to say, if Q is the inverse of P, and CQ meets the curve again in D, the conjugate of OT is represented by D. To verify this conclusion, we

observe that the angles between the tangents represented by C and D are halves of the angles between RC and RD, whence since the angles RCQ, RPC are equal, one of the angles between the tangents is MCP, already recognised as a conjugate angle of OT.

With the introduction of the point Q, there is a second representation of κ_n and ς_g and we can recognise the conjugate angle in another position, assuming that neither D nor E is zero, that is, that O is neither minimal nor umbilical. Since the angle RCQ is the complement of a conjugate angle, the conjugate angles are the angles between CQ and the tangent at C to the circle, and since by the elementary properties of inversion QC is $(E/D)PC$, the perpendicular distances of Q from the tangent at C and the radius RC are $(E/D)\kappa_n$ and $(E/D)\varsigma_g$; to render these results true algebraically and not merely numerically, it is necessary simply to regard RC as a directed line with direction from R to C, and the tangent as a directed line making a positive right angle with RC. It may be added that if OS, OT are any two *directions* on the surface, if B, C are the points of the circle representing the tangents with these directions, and if the chord BC is given the direction from B to C or the reverse direction according as the smallest angle from OS to OT is positive or negative, then the distance of Q from this chord measures algebraically the product by E/D of a function described in an unpublished paper communicated to the Society in February, 1916, and called there the bilinear curvature of the surface in the directions OS, OT.

Reference can now be made to a curve that I propose to call Euler's indicatrix; the equation

$$1/\rho_n = D - E\cos 2\xi$$

was actually given in other notation by Euler in his original memoir on the curvature of surfaces, and Euler illustrated the variation of normal curvature with direction by drawing the conic with polar equation

$$1/r = D - E\cos\theta.$$

It is hardly necessary to point out that Euler's indicatrix is the reciprocal polar of the circular indicatrix with respect to centre Q and constant E/D.

So far a point on the circle has been treated as representative of a *tangent* to the surface, with the result that references to angles on the surface have been somewhat loose. By duplicating the circle, in the manner familiar in function-theory, we can make a point on the double circle correspond to a *direcion* on the surface and discuss angles with precision. Possibly it would be an advantage in teaching to render stu-

dents familiar with duplication at an early stage by some such example as this.

Thursday, April 19th, 1917.

Prof. H. M. MACDONALD, President, in the Chair.

Present sixteen members.

Mr. G. N. Watson read a paper, "The Integral Formula for Generalised Legendre Functions."

Prof. H. Hilton read a paper, "On a Substitution Permutable with the Transposed Substitution."

Mr. E. H. Neville made a communication, "A Pair of Equations in the Theory of a Family of Surfaces."

An informal communication was made by Mr. G. H. Hardy "On the Theory of Probabilities."

In the absence of the author, the following paper was communicated by title from the chair :—

On a Liquid Gyrostat : Prof. W. Burnside.

ABSTRACTS.

The Integral Formula for Generalized Legendre Functions

Mr. G. N. WATSON.

In this paper I obtain a generalisation of a well known result in the form

$$\int_0^{(1+)} \{P_n^m(z)\}^2 dz = \frac{i \sin \pi m \sin \pi (n+m)}{2n+1} \frac{\Gamma(n+m+1)}{\Gamma(n-m+1}$$

$$\times \{\psi(\tfrac{1}{2}n-\tfrac{1}{2}m+1)-\psi(\tfrac{1}{2}n-\tfrac{1}{2}m+\tfrac{1}{2})+\psi(-\tfrac{1}{2}n-\tfrac{1}{2}m)-\psi(\tfrac{1}{2}-\tfrac{1}{2}n-\tfrac{1}{2}m)\},$$

where n and m are not restricted to be integers, but may be any numbers real or complex. The device employed in obtaining this formula is effective in evaluating several other integrals of a somewhat similar nature.

A Pair of Equations in the Theory of a Family of Surfaces

Mr. ERIC H. NEVILLE.

A paper on the differential invariants of space, read to the Royal Society by Forsyth in 1903,* concludes with the prediction of two relations then undiscovered between certain magnitudes connected with a regular function ϕ of position in space and the family of surfaces given by the equation

$$\phi = \text{constant}.$$

The theory of multilinear functions of direction described in an unpublished paper communicated to this Society last year† gives these relations immediately. They may be written as

$$\frac{d^3\phi}{ds_1^2 dn} - 2\left(\frac{d^2\phi}{ds_1 dn}\right)^2 \Big/ \left(\frac{d\phi}{dn}\right) + \kappa_{g1}\frac{d^2\phi}{ds_2 dn} + \left(\frac{d\kappa_{n1}}{dn} - \kappa_{n1}^2\right)\frac{d\phi}{dn} = 0,$$

$$\frac{d^3\phi}{ds_2^2 dn} - 2\left(\frac{d^2\phi}{ds_2 dn}\right)^2 \Big/ \left(\frac{d\phi}{dn}\right) - \kappa_{g2}\frac{d^2\phi}{ds_1 dn} + \left(\frac{d\kappa_{n2}}{dn} - \kappa_{n2}^2\right)\frac{d\phi}{dn} = 0,$$

where κ_{n1}, κ_{n2} are the principal curvatures of the surface, and κ_{g1}, κ_{g2} are the geodesic curvatures of the lines of curvature and are known to be given in terms of κ_{n1}, κ_{n2} by the equations

$$(\kappa_{n1} - \kappa_{n2})\kappa_{g1} = d\kappa_{n1}/ds_2, \qquad (\kappa_{n1} - \kappa_{n2})\kappa_{g2} = d\kappa_{n2}/ds_1.$$

It is only fair to add that the equations assume a more elegant form in the notation of the theory which leads to them.

Thursday, May 3rd, 1917.

Sir JOSEPH LARMOR, Vice-President, in the Chair.

Present also sixteen members.

Mr. G. H. Hardy read a paper, "Sir George Stokes and the Theory of Uniform Convergence."‡

* *Phil. Trans.*, A, Vol. 202, pp. 277–333.

† *Proc. London Math. Soc.*, Ser. 2, Vol. 15, p. xx.

‡ To be published in the *Proceedings of the Cambridge Philosophical Society.*

Prof. E. W. Hobson read a paper, "On Hellinger's Integral."

In the absence of the author, the following paper was communicated, by title, from the Chair :—

A Symmetrical Condition for Co-Apolar Triads on a Cubic Curve : Dr. W. P. Milne.

ADDENDUM.

*Note in addition to a paper "The Nodal Points of a Plane Sextic"**

Mr. J. HODGKINSON.

Prof. H. Hilton has pointed out to me that the questions treated in this paper had been considered previously by Halphen (*Bulletin de la Soc. Math. de France*, Vol. 10, 1882, p. 162). Halphen obtains substantially the same results by different methods, but finds a lower limit than mine (viz., 12 instead of 30, see § 5, p. 348) for the number of unicursal sextics which have nine given double points satisfying the necessary conditions. Halphen's estimate is most easily verified as follows. Suppose that a unicursal sextic S has double points at the nine fixed points $A, B, C, D, E, F, G, X, Y$; and that the tenth double point is Z. Then Y and Z lie on the locus ξ of the ninth degree associated with the eight points A, B, C, D, E, F, G, X; and X and Z lie on the locus η similarly associated with A, B, C, D, E, F, G, Y. The curves ξ, η intersect in eighty-one points. Each has a triple point† at each of the seven points A, B, C, D, E, F, G; and these points account for $7 \times 3 \times 3 = 63$ intersections. Also ξ has a triple point at X and passes through Y, and η has a triple point at Y and passes through X; so that X and Y account for $2 \times 3 = 6$ intersections. There remain $81 - 63 - 6 = 12$ possible positions of Z, and there are therefore at most twelve sextics of the type required.

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* *Proceedings*, Vol. 15, 1916, pp. 343–353.

† *L.c.*, p. 347.

Thursday, June 14th, 1917.

Prof. H. M. MACDONALD, President, in the Chair.

Present eighteen members.

The President announced the deaths of Sir William Niven (President, 1908–1909), and of Dr. W. H. Besant; he spoke briefly of their connexion with the Society and of their services to mathematical science.

The President announced that the Council had awarded the de Morgan medal for 1916–17 to Prof. W. H. Young, and that the medal would be presented at the Annual General Meeting, November 1st, 1917.

Mr. A. E. Jolliffe read a paper, "Some Theorems on the Multiplication of Series."

Mr. G. H. Hardy communicated a paper by Mr. S. Ramanujan, "On certain Trigonometrical Series and their Applications to the Theory of Numbers."*

In the absence of the author, the following paper was communicated, by title, from the chair :—

Examples of the Formal Analysation of Solutions of Differential Equations of certain Classes: Prof. E. B. Elliott.

ABSTRACT.

Examples of the Formal Analysation of Solutions of Differential Equations of certain Classes

Prof. E. B. ELLIOTT.

The scope of a method used on earlier occasions is enlarged, and two theorems of analysation are given by use of the identity among polynomials

$$1 = \sum_{r=1}^{r=m} \sum_{s=1}^{s=n} \left\{ \frac{1}{F'(a_r)\, G'(b_s)} \, \frac{F(\phi)\, G(\psi)}{(\phi - a_r)(\psi - b_s)} \right\},$$

where $$F(\phi) \equiv (\phi - a_1)(\phi - a_2) \ldots (\phi - a_m),$$

and $$G(\psi) \equiv (\psi - b_1)(\psi - b_2) \ldots (\psi - b_n).$$

In exemplification :—(1) the general solution of

$$\frac{\partial^{m+n} u}{\partial x^m \partial y^n} = f(x, y)$$

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* The paper will be published in the *Transactions of the Cambridge Philosophical Society*.

is expressed as a sum of mn parts each involving a double integral, the parts of a *given* solution u being determinate from it by direct operations, and (2) the general solution of a simultaneous pair of equations

$$\frac{\partial^m u}{\partial x^m} = f(x, y), \quad \frac{\partial^n u}{\partial y^n} = g(x, y),$$

for the consistency of which the condition $\frac{\partial^n f}{\partial y^n} = \frac{\partial^m g}{\partial x^m}$ is proved sufficient as well as necessary, is expressed as a sum of mn parts each involving one integral $\int (P\,dx + Q\,dy)$.

Finally, functions $f(x, y, z, z_{10}, z_{01}, z_{20}, z_{11}, z_{02}, \ldots)$, where z_{rs} means $\frac{\partial^{r+s} z}{\partial x^r \partial y^s}$, are analysed, and expressed as sums of directly determinate parts, which are exact partial derivatives of the various possible orders, in a way analogous to that used earlier* in developing a theory of exact derivatives.

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* *Messenger of Mathematics*, Vol. XLV, p. 33.

LIBRARY

Presents.

Between December 30th, 1916, and December 31st, 1917, the following presents were made to the Library by their respective authors and publishers :—

Blickfeldt, H. F.—"Finite Collineation Groups."
Cauchy, Augustin, Œuvres de, 2me série, vol. 12.
Dunk, John L.—"Hyperacoustics."
Jones, E. Lesler.—"Hypsometry, Use of Mean Sea Level as the Datum for Elevations."
Knibbs, G. H.—Appendix A, vol. 1, "Census of the Commonwealth of Australia."
Robbins, F.—"Factorials and Allied Products with their Logarithms."
Poincaré, Henri, Œuvres de, tome 2.

Amsterdam: Société Mathématique, Revue semestrielle des Publications Mathématiques, tome 25, partie 1.
London: Educational Times, nos. 661–663.
London: Educational Times, Math. Questions and Solutions, vol. 3, pts. 2–6; vol. 4, pts. 1–6.
London: Nautical Almanac for 1920.
Madras: Indian Mathematical Society, Journal, vol. 9, nos. 1–5.
Paris: L'enseignement mathématique, ann. 18, nos. 5, 6; ann. 19, nos. 1, 2, 3.
Sendai: Tôhoku University, Science Reports, vol. 6, nos. 1–3.
Sendai: Tôhoku Mathematical Journal, vol. 10, nos. 1–4; vol. 11, nos. 1–4; vol. 12, nos. 1–3.
Tokyo: Sūgaku-Butsurigaku, vol. 8, nos. 20–22; vol. 9, nos. 1–8, 10.
American Ephemeris and Nautical Almanac for 1919.
International Catalogue of Scientific Literature, 14th Annual Issue, (A) Mathematics and (C) Physics.

Exchanges.

American Journal of Mathematics, vol. 39, nos. 1–4.
American Mathematical Society, Bulletin, vol. 23, nos. 4–10; vol. 24, nos. 1–3; List of Officers and Members.
American Mathematical Society, Transactions, vol. 18.
American Philosophical Society, Proceedings, vol. 54, no. 220; vol. 55, nos. 3–8; vol. 56, nos. 1, 2.
Amsterdam: Nieuw Archiev, deel 12, stuk 3.
Amsterdam: Wiskundige Opgaven, deel 12, stuk 3, 4.
Baltimore: Proceedings of National Academy of Sciences, vol. 2, nos. 9–12; vol. 3, nos. 1–7.
Calcutta: Mathematical Society, Bulletin, vol. 6, 1914–15; vol. 7, 1915–16; vol. 8, no. 1.
Calcutta: Indian Association for the Advancement of Science, Proceedings, vol. 2; vol. 3, pt. 4; Report for 1915.
Cambridge Philosophical Society, Proceedings, vol. 19, pts. 1–3.
Cambridge, Mass.: Annals of Mathematics, vol. 18; vol. 19, no. 1.

Cape Town: South African Association for the Advancement of Science. The South African Journal of Science, vol. 12, no. 13; vol. 13, nos. 1–11; vol. 14, nos. 1, 2.
Catania: Accademia Giornia, Bollettino, fasc. 40, 41.
Chicago: John Crerar Library, 22nd Annual Report for 1916.
Copenhagen: Mathematisk-fysyske Meddelelser, vol. 1, nos. 1, 2.
Copenhagen: Oversigt Vid. Selskabs, 1916, nos. 4–6.
Copenhagen: Mémoires de l'Académie des Sciences et des Lettres de Danemark, Section des Sciences, 8me série, tome 1, nos. 4, 5; tome 2, no. 5.
Dublin: Royal Irish Academy, Proceedings, vol. 33, nos. 4, 5.
Edinburgh: Mathematical Society, Mathematical Notes, nos. 14–18, 21.
Edinburgh: Mathematical Society, Proceedings, vol. 35, pts. 1, 2.
Edinburgh: Royal Society, Proceedings, vol. 37, pt. 1.
Firenze: Biblioteca Nazionale Centrale di Firenze, Bollettino, nos. 187–190, 192–196, 198–204.
Holland, La Hague: Société des Sciences à Harlem, Archives Néerlandaises, vol. 4, no. 1.
Institute of Actuaries, Journal, vol. 50, pt. 4.
Institute of Naval Architects, Transactions, 1916.
Institution of Civil Engineers: List of Members, 1917.
Kyoto: College of Science, Memoirs, vol. 2, nos. 1, 2.
Livorno: Periodico di Matematica, anno 32, nos. 1–9.
Livorno: Periodico di Matematica, Supplemento, anno 20.
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OBITUARY NOTICES

SAMUEL BRUCE MACLAREN.

Samuel Bruce Maclaren was born on August 16th, 1876, at Yedo, in Japan. When he was about five years of age, his parents, who were both Scotch, removed to Melbourne, where his father, Samuel Gilfillan Maclaren, had been appointed Principal of a Presbyterian College. From his earliest years Maclaren showed great ability, carrying off the highest honours, both at his school, the Scotch College, Melbourne, and at the University of Melbourne. One of his teachers at the University stated in 1903 that Maclaren was by far the ablest student he had met during his twelve years' tenure of office, and one whose ability should be sufficient to place him in a very conspicuous position as an original thinker. None who knew Maclaren intimately could doubt the truth of this verdict, for few men gave so strong an impression of reserve power. Although there is evidence that the voluntary exile from his domestic ties, which in his case were strong, cost him much, he came to England in 1897, and entered Trinity College, Cambridge. He was elected into a Major Scholarship in 1899, being Third Wrangler in the same year, and took his degree in 1900. Taking Part 2 of the Mathematical Tripos in his third year, at that time an unusual course, he was placed in the Second Division of the First Class. In 1901 he was elected to an Isaac Newton Studentship, and graduated M.A. in 1905, proceeding about this time to the University of Birmingham as Lecturer in Mathematics.

At Cambridge he gave perhaps the impression rather of idleness than of industry, if judged by the quite fallacious standard of published papers. It was characteristic of Maclaren that the subjects which absorbed his attention most completely were always those of a really fundamental nature. He never published a paper dealing with any problem whose importance was merely secondary, or in any sense only a matter of detail. The nature of the work which he produced at a later period leaves no doubt that the more fundamental problems of mathematical physics were taking shape in his mind during the Cambridge period, and had been pondered unceasingly in the light of the newer theories

then springing up, which appeared to be undermining the whole foundation of the theory of radiation as ordinarily conceived. The opportunity of collecting his work into a connected scheme was presented to him when the subject of the Adams Prize Award for 1913 was announced. He was successful in this competition, and one of the tragedies of his untimely death is the fact that it occurred under circumstances which had precluded him from publishing his essay. It is not at present certain how much of it is in being, for he was occupied in its revision immediately before the outbreak of war. Much of his work may be lost irretrievably, and buried for ever in his grave at Abbéville. But even when our attention is confined, as perforce it must be, to the few papers which he actually published, his contribution towards the problem of radiation and of the nature of matter is found to be very considerable, and it is pervaded by a highly original point of view developed with great mathematical skill.

In 1913 Maclaren was appointed to the Chair of Mathematics at University College, Reading, and, in the brief period before the outbreak of war, he threw himself completely into all the activities of a rising institution seeking an existence as an independent University. His personal popularity among his colleagues was very great, and many tributes have been paid by them to his memory. Previously somewhat aloof from the scientific societies whose scope included his own main interests, during this period he became more active in this regard. At the time of his death, he was a Secretary of Section A of the British Association, and a peculiarly appropriate officer in view of the Australian visit. There is no doubt that he keenly appreciated this opportunity of returning to familiar scenes, marred, however, by the death of his father. The greatness of the shock which this was to him was known only to his more intimate friends.

This is not the appropriate place in which to give any full account of his work. His manuscripts are in the hands of those who have undertaken the task of saving as much as possible from oblivion, and a few general remarks are alone desirable, relating entirely to his published papers. These may be found in the *Philosophical Magazine* for 1911 and later years. In the first important paper, Maclaren discusses the emission and absorption of energy by electrons, and, in a very original manner, extends Lorentz' theory of complete radiation to all wave-lengths. The radiation is regarded as an external field superposed on the other forces which act on the electrons, and Maclaren points insistently to the essential fact that we must be at liberty to treat the radiation as a small disturbing force which gradually takes the electron out of the orbit it would follow under the internal forces alone. When this is ensured, we can use

the theory of varied motion, the displacement due to radiation being small during the time of describing a free path, small in comparison with length of path and with any wave-length under consideration. Maclaren takes this as the key method in all his investigations, and develops it with considerable power. In this first paper, he points out that the exponential term in Wien's law belongs to the emission, and the paper concludes with a general formula, of the type of an integral equation, for the partition of energy in wave-length, to which any assigned distribution of electronic velocities can be fitted, and giving the Lorentz formula with the kinetic theory of distribution.

These ideas are developed further in a second paper, published a few months later. When the problem of energy distribution is thus reduced to the determination of a function f, it is found that the determination is beyond the reach of our present dynamical theory. If Hamilton's equations as derived from Least Action, and an essentially positive energy function, are used, we must have equipartition. We can give up Least Action and alter Hamilton's equations without coming nearer to a solution of the radiation problem, for no continuous laws of motion in which energy is conserved can account for the distribution of energy realised in nature, and the revolution foreshadowed by Planck and Einstein appears to become inevitable.

Such conclusions were, of course, reached by Jeans and Poincaré also, but Maclaren's investigation proceeded on quite different lines. He did not like to assume that the statistical method is applicable to the æther with its infinity of degrees of freedom, and accordingly undertook, in a later paper, a direct calculation of emission and absorption, applicable to matter obeying any continuous laws of motion whatever. This paper is an attempt to save the classical view of radiation as a continuous wave motion. Maclaren regarded it as a relatively small thing to sacrifice merely the ordinary mechanical notions of matter, being prepared to accept atomism for matter but not for radiation. In this paper he assumes the undulatory theory of light, but not the formulæ of classical dynamics. The laws of motion of matter need not now be continuous laws at all, though some at least of the variables specifying the conditions of matter must have a continuous range. Discontinuous changes must be governed by conditions not directly dependent on the presence of radiation, and, for example, may occur when velocities or coordinates reach prescribed values. Equipartition ensues as usual if the Principle of Least Action be assumed, but in the absence of this assumption, a formula very similar to Planck's can be derived on certain simple alternative hypotheses, although, as Maclaren states, nothing is done to show that these hy-

potheses are actually consistent with possible laws of motion. The whole investigation is highly interesting and suggestive, and quite unlike anything to be found in other treatments of this question. Maclaren at the same time shows that the classical dynamics cannot furnish an adequate kinetic theory of induced magnetisation.

His last published paper is in many respects the most striking, and begins by resuming, from a new standpoint, the discussion of a theory of gravity which goes back to Riemann, of æther as fluid, and of matter as a place of destruction, gravity being an interference effect of the æthereal motions associated with portions of matter. The paper is very long, and an abstract cannot be attempted here. But Maclaren passes on to the modern theory usually connected with Einstein, or rather to a theory independently derived, and of a similar nature. He was a believer in the physical reality of the fourth dimension, and went further than Minkowski. Though, as with Minkowski, the four-dimensional universe is without change, it is not, with Maclaren, without time or motion. Time appears as a purely logical succession, the symbol of an order in which the elements of Minkowski's space are throughout.

Maclaren presented some of the more fundamental parts of his work in this connection to the Mathematical Congress at Cambridge in 1912, and his results were in accordance with those of Abraham, and obtained independently by a quite different mode of procedure. He undoubtedly anticipated Einstein and Abraham in their suggestion of a variable velocity of light, with the consequent expressions for the energy and momentum of the gravitational field, and put forward, in his last paper, a suggestion, perhaps of great value, as to the nature of the magneton and of the elementary positive charge.

Maclaren was singularly modest regarding his work, and this trait, combined with a great number of other interests, and a personality which was even boyish to the casual observer, tended to hide his real gifts from all but his more intimate friends and those familiar with his writings. On all the matters to which he gave greatest thought he was one of the most reserved of men. He had the highest sense of duty, and few know the mental crisis through which he went while in Australia with the British Association at the outbreak of war. He studied signalling on his way home, and having reached his decision, hesitated no further. We must admire such a decision in one for whom the military life had no attraction, but at the same time regret a national state of mind which could not suggest, to a man of this type, that the best national work he could do was in a different category. The parallel between his case and that of Moseley is very close, and they are perhaps the two most irre-

parable losses to the future of British science which this War has produced.

Maclaren was, as an officer, idolised by his men, and was conspicuous for personal bravery in an army where bravery is almost universal. He met his death as the result of an endeavour to save a store of bombs threatened by fire. Though already badly wounded, and begged to desist, he went on and received the further wounds from which he died a few days later, on August 13th, 1916, at Abbéville. The bombs were not saved, and his own life, and with it much that future generations might have hoped for, was lost.

J. W. N.

SIR WILLIAM DAVIDSON NIVEN.

WILLIAM DAVIDSON NIVEN was born at Peterhead in 1843, and died at his residence in Sidcup in May 1917. After a brilliant career at the University of Aberdeen, he passed on in 1862 to Trinity College, Cambridge, of which he became a Fellow in 1867. In the Mathematical Tripos of January 1866 he had come out as third wrangler, bracketed with his friend James Stuart of Trinity, who was soon to be prominent in the starting of the system of University extension. Later, in 1875, Stuart was elected, by the suffrages of the Senate at large, to a newly founded chair of Mechanism and Applied Mechanics—the nucleus of the present School of Engineering—which he held until 1890. Subsequently he devoted himself in a Parliamentary career mainly to political and social problems, and after some years of retirement and occupation with business affairs, predeceased Niven only by a short time. The senior wrangler of the year was another Scotsman, Morton of Peterhouse, who, coming from the University of Glasgow, was in succession a Fellow of that College and of Christ's, but died young.

It was a time of distinction in the Mathematical Tripos. In the previous year the first two wranglers were Lord Rayleigh and the economist Alfred Marshall. In the following year his brother Charles Niven, now Professor of Natural Philosophy at Aberdeen, was Senior, the mathematician and philosopher W. K. Clifford was second, and Osborne Reynolds, physicist and engineer, was seventh: while in the next year Lord Moulton was senior, Sir George Darwin, late Plumian Professor of Astronomy, second, and Sir W. Christie, formerly Astronomer Royal, fourth.

A third brother James Niven, eighth wrangler in 1874, became Fellow of Queens'; and, having graduated M.B., is now head of the Department of Public Health for Manchester, and a well-known authority on that subject.

Soon after obtaining his fellowship at Trinity, W. D. Niven took up the appointment of Professor of Mathematics at the Royal Indian Engineering College, Coopers Hill. This he presently exchanged for the probably more congenial Chair directing the Mathematical work of the advanced class of Artillery Officers at Woolwich; here he came into practical contact with the rapidly advancing science of gunnery and ballistics, and initiated improvements in the theory of trajectories which maintained their

place for many years, until at length the modern high velocities had made the trajectories so flat as to greatly simplify the practical calculations. The new developments of the present war have probably reinstated, and doubtless improved on, the older investigations.

In 1874 Niven accepted an invitation to return to Cambridge, as a lecturer on the staff of Trinity. Here he came in contact with Clerk Maxwell, who had been persuaded by scientific pressure to come back to the University in 1871 as the first Cavendish Professor of experimental physics; the Cavendish laboratory, presented by the Chancellor, the Duke of Devonshire, was completed under his direction in 1874. A close friendship seems to have struck up at once between the two men, congenial in every way, which went on with increasing force until Maxwell's untimely death in 1879 at 48 years of age. Niven charged himself largely with winding up the scientific affairs of his friend: the main formal memorial, two volumes of Scientific Papers, of quarto size now inconveniently large, was prefaced with an introductory biography which his busy life after leaving Cambridge too long delayed. He also completed for press the second edition of the *Electricity and Magnetism*. This period of his work in Cambridge was marked by a great expansion of the mathematical studies of the place in the direction of physical science. As was natural, the new developments of Maxwell in the theory of electricity excited keen interest, at a time when they were largely misunderstood or not understood at all elsewhere. The public lectures of Niven were on the subject of electricity and magnetism; and his class-room was a focus for the theory, in which congregated practically all the mathematicians of the University. The relations thus formally begun were never allowed by him to drop; and in later years he could count nearly all the active developers of electrical science on the mathematical side as his friends and former pupils.

In 1882 Niven was again tempted to leave Cambridge, in order to take up the important office of Director of Studies at the Royal Naval College, Greenwich, which involved the general direction and supervision of education in connection with the Royal Navy. His talent for friendship and quiet social amenities made his beautiful residence, on the river front of the Palace of Greenwich, a centre of the life of the staff; and here also he won and retained through life the friendship of successive admirals who came for a term of office as Governors of the College. He induced some of his Cambridge friends of high mathematical eminence to undertake posts on the staff: by his hospitality he made the position of external examiner so attractive that he could command the services of the best qualified men of the time. On his retirement at the age of sixty in 1903

his official services were recognised by the conferment, then very unusual, of a Knight Commandership of the Bath. By his continued interest in the scientific societies, notably the Royal Society and the London Mathematical Society, he became a considerable stimulus on the social side to the activity and intercourse of scientific men.

From 1877 he devoted sustained attention (writing mainly in the *Messenger of Mathematics*) to the development of Maxwell's beautiful method expounded in the *Electricity and Magnetism* (1872), perhaps adapted from the treatment in Thomson and Tait's *Natural Philosophy* (1868), of defining the general spherical harmonic in terms of its poles. The expression

$$(-1)^i \frac{d^i}{dh_1 dh_2 \dots dh_i} \frac{1}{r}, \text{ or shortly } (-1)^i \left[\frac{d}{dh}\right]^i \frac{1}{r},$$

in which r is the distance of a point from the origin, and δh_1, δh_2, ... are displacements of the origin in the directions of the poles, can represent any solid harmonic of degree $-(i+1)$, and also, when multiplied by r^{2i+1}, its conjugate of degree i. Niven proves the theorem, and makes it fundamental, that

$$(-1)^i \left[\frac{d}{dh}\right]^i \frac{1}{r} = \frac{1}{r^{i+1}} \left[\frac{d}{dh}\right]^i Q_i r^i,$$

where $Q_i(\mu)$ is the zonal surface harmonic of order i having r as its axis, and $r\mu = z$. He employs this theorem to obtain rapidly Maxwell's development of the general harmonic in terms of variables such as μ_s representing $\cos(rs)$ and μ_{st} representing $\cos(st)$, where s, t denote vectors to the poles: thus for the first few orders

$$Y_1 = \mu_1, \quad 2!\, Y_2 = 3\mu_1\mu_2 - \mu_{12}, \quad 3!\, Y_3 = 3(5\mu_1\mu_2\mu_3 - \mu_{23}\mu_1 - \mu_{31}\mu_2 - \mu_{12}\mu_3),$$

which take of course the well-known zonal forms when the poles all coincide with the axis, and the tesseral forms when some of them coincide with the axis and the others are equally distributed around its equator. This method is illustrated by application to Laplace's expansion of a function in a series of harmonics.

Incidentally he notes that

$$1.3.5 \dots (2n-1) \left[\frac{d}{dh}\right] \frac{1}{r^{2n+1}} = (-1)^i \frac{1}{r^{2n+i+1}} \left[\frac{d}{dh}\right]^i \left(r^i \frac{d^n}{d\mu^n} Q_{n+i}\right),$$

which points to wide though not relevant analytical extensions.

It had been formally established by Sylvester (*Phil. Mag.*, 1876), from simple considerations of geometrical representation, that the system of poles which prescribes a given solid harmonic is unique. It may be re-

marked that a harmonic may have uniform relations with a set of axes, without those axes being its polar axes: for two or more poles may coincide, and any symmetric function of the polar operators will generate such a harmonic: in this way the harmonics possessing the symmetry of regular or other solids might be expressed in a direct manner.

In a following paper he notes that any finite function V can be expressed in terms of gradients of various orders of its value V_0 at the origin, by the formula

$$V = \exp\left(x\frac{d}{dx}+y\frac{d}{dy}+z\frac{d}{dz}\right)V_0 = \Sigma\,\frac{1}{i!}\left(x\frac{d}{dx}+y\frac{d}{dy}+z\frac{d}{dz}\right)^i V_0,$$

in which, when V is r^{-1}, the term of order i represents the general solid harmonic of that degree: and he employs a symbolic analysis based on this formula to evaluate directly the various surface integrals of products of harmonics over a sphere.

These papers were preparatory to a very elegant memoir of high analytical power, *Phil. Trans.*, May 1879, pp. 379–416, in which he exhibits the scope of this method by various applications, especially to the evaluation of the integrals over a sphere of products of three or more harmonics, and of products of them with other functions, a subject in which some striking results had previously been elaborated by laborious special methods. The symbolic exponential expressions led him directly to the result from which these developments flow by expansion, in the form that, if V is any function expressible within the sphere of radius R by a Taylor series, then over that sphere

$$\iint V\,dS = 4\pi R^2 \sum_0^\infty \frac{R^{2i}}{2i+1!}\left(\frac{d^2}{dx^2}+\frac{d^2}{dy^2}+\frac{d^2}{dz^2}\right)^i V_0,$$

where the operators refer to the value V_0 at the centre. At the end the method is extended to the evaluation of integrals over an ellipsoid, including the potentials of ellipsoids and circular electric currents, starting from the related expression for $\iint Vp\,dS$.

He returned to the subject, and generalised it much further, in October 1890, in a memoir "On Ellipsoidal Harmonics", *Phil. Trans.*, 1891, pp. 231–278, stimulated by reading Green's Memoir on Ellipsoids of varying densities; and his work retains some of the direct and forceful power of its source of inspiration. The adaptation of the analysis, including a correlation between spherical and internal ellipsoidal solid harmonics, starts with the relation, adapted from Maxwell, that if $f(x, y, z)$

and $F(x, y, z)$ are two solid spherical harmonics of degree n then

$$\iint f(x, y, z)\, F(x, y, z)\, dS = 4\pi\, \frac{2^n n!}{2n+1!}\, f\left(\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}\right) F(x, y, z),$$

and proceeds from the general function expressible in a Taylor series as before. External ellipsoidal harmonics are treated as differential derivates from the expression for the potential of a uniform ellipsoid. The formulæ are developed in detail for cases of axial symmetry: also for the cases of cylindrical and paraboloidal harmonics. The fundamental harmonic expression for the reciprocal of the distance between two points is developed. Many expressions giving solutions of physical problems occur incidentally.

The electrostatic capacity of a lens-shaped conductor is readily expressed in finite terms as an example of Kelvin's theory of electric images, when the internal angle α at which the bounding spherical surfaces meet is $2\pi/n$, where n is integral. For other values of α the capacity must therefore be equal to this expression together with some function $lF(\alpha)$, where l represents the linear scale of the system, and $F(\alpha)$ has the property of vanishing when $\alpha = 2\pi/n$ and so has roots which become infinitely numerous near the value zero. A property of this type is not unusual: the function $\sin \alpha^{-1}$ is a case in point.

The problem of determining the distribution of free charge, and the capacity, for general values of α was attacked by Niven in two papers (*Proc. London Math. Soc.*, 1894 and 1896) employing a method remarkable as being purely abstract. He constructs a functional equation which the potential must satisfy, somewhat after the manner of Poisson's discussion of the electrostatic distribution on two adjacent complete spheres, and effects a solution, subject to conditions imposed by the problem, in the form of a definite integral. This unusual procedure excited considerable interest and perhaps some hesitation as to the sufficiency of the argument. When H. M. Macdonald attacked the problem by the standard method of Bessel functions, he reached results at first sight different; but he soon had the satisfaction of establishing their identity with those of Niven. The results, of course, were a considerable enlargement of the few previously known solutions for a conductor partially enclosing a hollow space: for they extended Lord Kelvin's classical case of a thin spherical bowl to a bowl bounded by segments of any two intersecting spheres. A striking corollary, easily verified for special cases, is that the difference of the charges on the two faces of the bowl is equal to the potential multiplied by a differential capacity

$$a\, \frac{\pi}{\gamma}\, \cot\, \frac{\pi}{\gamma}\, a,$$

when the two surfaces intersect at an interior angle α, and one of them intersects the plane of the circular edge, whose radius is a, at an interior angle γ. Previous papers in *Proc. Math. Soc.* were a reconstruction of the analytical theory of electric images applied to two influencing charged spheres (Vol. VIII, 1876, pp. 64–83), and an extension of the same method to intersecting spheres (Vol. XII, 1880, pp. 27–36). There is also an elaborate paper on the harmonics of an anchor ring (Vol. XXIV, 1893), in which the expressions are built up from a system of infinitely thin rings, and applied to the electrostatic influence between two solid rings, or a sphere and a ring, on a common axis.

For many years W. D. Niven was one of the chief supporters of the London Mathematical Society, serving on the Council for fourteen years, at various times beginning with the year after his election into the Society in 1873. He was President for the Sessions 1908 and 1909. On retiring from the chair in November 1910 he delivered the customary address, choosing as his subject "The Relations of Mathematics to Experimental Science": with characteristic modesty he seems to have withdrawn it from publication.

He was elected into the Royal Society in 1882, and was on its Council for the periods of 1892–4 and 1904–6, serving as Vice-President for the latter two years.

A few years ago his scientific friends and past pupils, of Cambridge, London, and Aberdeen, combined to have his portrait painted in oils. It was presented to the University of Aberdeen, where it hangs in King's College.

J. L.

JEAN GASTON DARBOUX.

JEAN GASTON DARBOUX, one of the most illustrious of our honorary members, was born* at Nîmes on the 13th of August, 1842, and died in Paris on the 23rd of February, 1917. His father died when the boy was seven years old; his immediately succeeding years were sedulously directed by his mother, who had the intelligence and the outlook that are characteristic of so many French women. Seeing that the boy's disposition tended towards study rather than business, she made every effort to secure for him a steady, continuous education.

First at Nîmes, and later at Montpellier, he pursued his studies, with an obvious bent towards mathematics. At the age of nineteen he came out at the head of the list for admission into the École Polytechnique, and the École Normale Supérieure in the section of Science. The choice of school rested with him; so he chose the École Normale. The selection was a surprise at the time; for it meant that the candidate elected for teaching, instead of the more honoured activities of public service as then estimated; and it had never previously been exercised in that sense. But the choice was applauded by Pasteur, the sub-director of the school; later, it was followed as a precedent by Appell, Picard, and others; and, fifty years afterwards, it still appealed to the imagination of his colleagues.

At the École Normale, he spent three years as a student. He attended the lectures of Bertrand, thus beginning a friendship that suffered no break until the death of the older man; and other friendships then were formed with men whose names now are classics, Briot, Bouquet, Chasles, Serret. His main studies appear to have been in the range of geometry. His first original paper was concerned with orthogonal surfaces, and was presented to the Academy of Sciences by Serret in 1864. In that year he graduated, again at the head of the list. Pasteur, now the director of the scientific section of the school, had a post specially created for him, yielding him enough leisure for original work. The result appeared in his thesis, *Sur les surfaces orthogonales*, by which, in 1866, he obtained his doctor's degree.

For the next six years he was occupied in secondary teaching, which proved rather exacting. Still, he found time to pursue his researches in geometry; and he began to publish short memoirs on other subjects, such

* The writer of this notice is indebted, for many of the biographical facts, to the volume *Éloges académiques et discours*, published on the occasion of Darboux's jubilee, and to a monograph *Gaston Darboux*, by M. Ernest Lebon. [These pages were in type before the writer had seen the *Notice historique*, by M. Émile Picard, read at the meeting of the Academy of Sciences on 10th December, 1917.]

as the singular solutions of ordinary differential equations of the first order, and the theory of partial equations of the second order.

One personal incident of this period may be recalled. At the outbreak of the Franco-Prussian War, in 1870, Sophus Lie was one of the foreign students attending Jordan's lectures in Paris. Being a Norwegian, he was allowed to remain if he wished. He determined on a long tramp to Italy. But he went no further than Fontainebleau. There he wandered among the woods and busied himself with his calculations and geometrical figures. He was soon arrested and imprisoned by military authorities, who saw spying and plans of fortresses in the names of mathematicians and in diagrams of complexes. After several unsuccessful attempts at communicating with French mathematicians, who were absent on the national defence, he got into relation with Darboux; and Darboux's efforts won freedom for Lie.

In 1872, Darboux passed from secondary teaching to work of a University standard, returning in the first instance to the École Normale. From that date, his promotions were regular and rapid. After having filled intermediate chairs, he was appointed, in 1881, to the Professorship of Higher Geometry in the Faculty of Sciences at Paris, a chair originally created for Chasles in 1846. Not an unworthy successor of that illustrious geometrician even at the date of his appointment, he continued his researches and developed his teaching for some thirty years; and long before the end of that period he would, I think, have been acclaimed as the greatest living master of differential geometry.

In 1884 he was elected a member of the Academy of Sciences in the section of geometry. He was made Dean of the Faculty of Sciences in 1889 and held that office until 1903, when, on his resignation, he was succeeded by Appell. In 1900, on the death of Bertrand, he was appointed Perpetual Secretary of the Academy, on the side of the mathematical sciences. He discharged his varied duties with a zeal, a success, and a courtesy that were universally acknowledged; and he held this high and responsible office until his death.

He had entered the profession of teaching by entering the École Normale in 1861. Fifty years afterwards there was a celebration of this event, in which mathematicians from all over the world took part.

He was a not infrequent visitor to England, mostly for important ceremonials or for meetings of international committees. Thus he was one of the leading members of the Central Committee of the International Association of Academies. At the Stokes Jubilee, in 1899, he was the representative of the University of Paris; as that University had the pride of age among all the bodies represented, Darboux had the pride of place. And at the funeral of Lord Kelvin in Westminster Abbey, in 1907, he came as the representative of the Institute of France.

He spent his life within what may be called an academic round; and it obeyed the Horatian injunction

Servetur ad imum,
Qualis ab incepto processerit, et sibi constet.

As a teacher, an investigator, and an administrator in many fields, he had a full life. Academic honours, the most fitting recognition of such work, came to him in profusion. He was a foreign or an honorary member of more than thirty Academies and learned Societies. It is a satisfaction to note that our own Society was one of the very earliest to pay tribute to his genius; he was made one of our honorary members as early as 1878, and so was nearly the senior among them at the time of his death. He was elected a foreign member of the Royal Society in 1902, and, in 1916, was awarded the Sylvester medal. And the University of Cambridge in 1899 conferred upon him an honorary Doctorate of Science on the occasion of the Stokes Jubilee.

He had the reputation of being an ideal professor, alike in teaching and in stimulus; in so far as his books and his memoirs give indication of professorial qualities, he certainly deserved his reputation. He was considered excellent as an administrator, alike in organisation and in development, and possessed "business capacity" which is not supposed to be the possession of all able mathematicians. The writer of this notice remembers well the easy strength with which Darboux presided over the meetings of the International Association of Academies in its first general assembly at Paris in 1901; yet there were some fiery members among the delegates. Time after time, in earlier years as a representative of the Faculty of Sciences, and in later years as a Perpetual Secretary of the Academy, it was his lot to deliver historical and other addresses, and to write commemorative notices of distinguished men of science. Darboux, in full measure, had the power of writing just appreciations, and, above all, he had that grace of style which is a very special possession of French writers.

Darboux was quite tall, of good presence, and (as he struck me) inclined to be thin. He wore *pince-nez*, and, at a distance, his face seemed to be a little severe; but when he recognised an acquaintance, the severity melted at once into the sunlight of a warm and friendly welcome. He had a quality of humour which revealed itself in addresses and in speeches, though not in his mathematical work; most mathematicians (Maxwell was an exception) will not allow a flavour of humour in their published work.

Darboux's original contributions to mathematical science covered a large field. In the later years of the last century, and in the earlier years of the present century, he would probably have been known best by his work in geometry, especially infinitesimal geometry. There his influence was supreme, through his memoirs and books, and through the work of a

succession of brilliant pupils inspired by him. But in his life he made valuable contributions to many other subjects : to analytical mechanics, celestial mechanics, and mathematical physics, to algebra and algebraical geometry, to several branches of the integral calculus and the theory of functions, and to parts of the theory of differential equations. In illustration of these statements, a few of his memoirs will be mentioned briefly.

Many of his early papers were concerned with applied mathematics ; and their topics doubtless arose in connection with his duty to teach that subject at the Sorbonne in the early seventies. These were partly summarised in a set of notes which he added in an edition (1889) of Despeyrous's *Cours de Mécanique*, in his two-volume edition (1888, 1900) of Fourier's works, and in his edition (1888) of Lagrange's *Mécanique Analytique*.

In the integral calculus, two publications stand as of special importance. One of these is his monograph (1882) on Pfaff's Problem ; the substance was communicated (1876) to Bertrand, who incorporated it in his lectures in the Collège de France. He dealt with the problem as belonging to the theory of equivalent forms, and hardly at all as connected with the corresponding integral equations ; and the derivation of his results was made to depend upon the significance of certain invariants of the differential form. The other memoir (1874), to which reference is made, was devoted to discontinuous functions ; in it he resumed and extended Riemann's discussion of definite integrals. Much further progress in this last subject has been made in the last twenty years ; but Darboux's contribution was a real development at the date of its publication.

In the region of differential equations, he made advances that were at once notable and useful. In the late sixties, much had been written round the singular solutions of ordinary equations of the first order ; and not all of it was satisfactory. Cayley in 1872, and Darboux very early in 1873, working independently of one another, published some investigations, brief in extent, which have remained a stable foundation for the theory of those singular solutions. Their results have been greatly amplified ; the publication was an epoch in the theory.

A much more extensive and impressive work was Darboux's crowned *Mémoire sur les solutions singulières des équations aux dérivées partielles du premier ordre* (1880). In its wide range and its novel results, and specially in its development of the theory of characteristics and its applications of contact transformations, it is little less than a treatise on the theory ; and it holds a prominent place in the subject.

Partial equations of the first order can be regarded as integrable : that is to say, the difficulties which occur in constructing the integral are merely difficulties of manipulative analysis, such as quadratures and the integration of ordinary equations ; they are below the domain of the theory of partial equations. But it is a vastly different matter when we have to deal

with partial equations of the second order (and, as a natural consequence, of order higher than the second). There is a great array of literature dealing with some important individual equations of the second order, especially those which occur in mathematical physics; but methods of general integration are sadly lacking. The name of Mouge, of course, occurs at once; success with his method is met only for very particular types of equations, and the conditions for success have been formulated. The method of Ampère is more general as regards types of equations to which it can be applied effectively; but it is difficult, and most workers soon find themselves stranded when applying it to all but the simplest equations. Darboux tackled the problem from another side, dealing with equations that are integrable in a form that is free from what Ampère called "partial quadratures". The idea of Charpit, for equations of the first order with two independent variables (ultimately extended by Jacobi to equations of the first order with any number of independent variables), was to associate with a given equation of the first order an independent but consistent equation, also of that order. Darboux amplified the notion by associating with an equation of the second order in two independent variables an independent and consistent equation, it might be of the first order, or of the second order, or of a higher order; and his method will provide a consistent associable equation of finite order if the original equation is of the Ampère type. His method, without supplying the complete solution, which usually cannot be obtained in the finite terms so dear to those who "solve" equations, marks a real advance in the theory; and it has led to wider investigations since its publication in 1870.

But, after all (and I am not attempting an exhaustive record of Darboux's work), he probably was best known to the contemporary world of mathematicians by his classical treatise on infinitesimal and differential geometry—a subject in which he was a veritable pioneer. As already stated, his earliest paper was on orthogonal surfaces, published in 1864; it was only in 1910 that he issued the completed form of his investigations on the subject in the volume, *Leçons sur les systèmes orthogonaux et les coordonnées curvilignes*. Between those two dates, the monumental treatise, *Leçons sur la théorie générale des surfaces et les applications géométriques du calcul infinitésimal*, appeared in four volumes, in 1887,* in 1889, in 1894, and in 1896 respectively. It is practically an encyclopædia of the subject, with a tremendous store of results, either new or newly developed; together with applications to and from all sorts of branches of mathematics, as varied as dynamics, soap-bubbles, partial differential equations, conformation, deformation, congruences, and cyclical systems. It is a great work, the achievement of a master.

* A second (and amplified) edition of the first volume appeared in 1914.

Mention should also be made of an exceedingly interesting volume, *Principes de géométrie analytique*. It practically contains his lectures on principles, delivered from time to time through many years at the École Normale and the Faculté des Sciences; it does not aim at being other than fairly elementary; and, in its union of analysis with what Poincaré called the "geometrical sense", it is characteristic of Darboux in his mental attitude towards geometry. It is probably the last work completed by him; for its preface bears a date only a few weeks earlier than that of his death.

Of his multifarious activities, only one other will find mention here. In 1870, he founded the *Bulletin des sciences mathématiques et astronomiques*. With a slight change of title in 1885, it has continued, to the great advantage alike of original papers and of reviews. He secured plenty of collaboration; Picard and J. Tannery were associated with him from the first. But the most prolific contributor must have been himself; he wrote over seventy articles in all, original papers, reviews of collected works and individual treatises and memoirs, indeed anything and everything that comes the way of an active editor.

If a mathematician's fame is to live, it must live on his enduring contributions to his science; it will hardly survive long from his teaching, and not at all from his administrative work, however useful. The judgment, as well as the admiration of his contemporaries, would predict that Darboux's original work will keep his fame fresh in long remembrance.

16th January, 1918. A. R. F.

WILLIAM HENRY BESANT.

WILLIAM HENRY BESANT was born at Portsea, Hants, on the first of November, 1828. He was the son of William Besant, of Portsea. He received his early education at the Grammar School at Portsea and at a Proprietary School at Southsea. He came up to St. John's College, Cambridge, as a Sizar, at the age of seventeen, and was Senior Wrangler and First Smith's Prizeman in 1850, when he was twenty-one. The popular coach and the maker of Senior Wranglers in those days was Hopkins, but Besant was coached by Stephen Parkinson, a member of his own college. Judged by modern standards, his knowledge of mathematics on entering the University was probably not extensive, for there is among his papers a small bound note book containing the notes that he made as an undergraduate, in 1847, of "Lectures on Trigonometry", by Mr. Griffin, and of "Sadleirian Lectures, 2nd Term" on Algebra, by Mr. Blackall; which shows the kind of work that was expected of a freshman reading for the Mathematical Tripos in those days. The notes of each lecture take the form of from six to ten questions, to some of which the solutions are given. The Trigonometry lectures evidently assumed little or no previous knowledge of the subject, and begin with the definition of an angle; but the Algebra course, in its second term, was more advanced, and included some elements of continued fractions and the theory of probability.*

Besant was elected to a Fellowship in 1851, and was appointed Lecturer in Mathematics at St. John's, in 1853, an office which he held for 35 years. He vacated his Fellowship on his marriage, in 1859, but was re-elected in 1889. He acted as Moderator and Examiner for the Mathematical Tripos in 1856, 1857, and 1885. He examined for the Smith's Prizes in 1859, was Examiner in the University of London for the five years 1860–64; and for many years examined in the Indian Civil Service Competition. He was elected a Fellow of the Royal Society in 1871, and, along with Dr. Routh, was one of the first Members of the University

* It is interesting to record that some years earlier it was possible to obtain a high place in the Tripos, starting as a freshman, with very little previous knowledge of mathematics. The late William Walton,† who was Eighth Wrangler in 1836, once told the writer that he only began Euclid when he entered the University.

to take the newly instituted degree of Sc.D. in 1883. He held the office of Esquire Bedell from 1866 to 1870.

But it is as a private tutor that Dr. Besant will be best remembered. For many years he shared with Dr. Routh deserved popularity as a coach, and won the esteem and affectionate regard of many generations of pupils. He was an indefatigable worker, at times beginning his daily round with pupils at 7 a.m. His method of teaching consisted for the most part in getting his pupils to copy manuscripts which he had prepared, and to write answers to series of bookwork and rider papers on all the subjects of the Tripos—papers of which he kept a few copies only in manuscript. He also set a weekly problem paper, and it was his habit to look over all the written work of his pupils between the coachings, which took place three times a week, before discussing it with them. As the number of his pupils must often have been as many as fifty or sixty he could not, but for the fact that he was a very rapid worker, have kept pace with all that he had to do. He was equally at home with all branches of mathematics, and always ready to discuss any sort of difficulty and to encourage his pupils to develop new ideas.

In the days of the undivided Tripos the skill of a coach lay in guiding his men to the best use of their time, as it was not possible for them to "read" or know all the subjects in the schedules, and to guide them wisely needed great judgment.

Besant's method as a lecturer in the early seventies was to dictate some ten or twelve problems from recent examination papers, and then after an interval go round and see how the class were getting on. He never solved anything for a pupil himself, but only suggested a start, or pointed out an error. He was always anxious that men should try geometrical as opposed to analytical methods, and he used to impress upon his class the usefulness of elementary subjects from the point of view of the examination. He was always kind and urbane, and, if you had a difficulty which you could explain to him, he always helped you, so that, in days when College lectures were of no great account, Besant's stood out as among the best.*

He wrote a number of textbooks, some of which are still in use, though more than fifty years have passed since they first appeared. Some of these were put together in a few weeks while on a holiday; this was so, I believe, with his first book, the *Hydromechanics*, which was first published in 1859. His most original work, which showed his great powers as a Geometer, was the *Notes on Roulettes and Glissettes*, which came out in 1870, probably the only mathematical book that has ever received a notice in *Punch*. Of this fact Besant was justly proud, for the work was his own, even to the word

* For the facts in this paragraph, and for some other details, I am indebted to the kindness of the Master of St. John's College.

"Glissette", which he invented, and he had the cutting from *Punch** pasted into his own copy. Besant always interested himself in the nomenclature of mathematics. The word "orthocentre" was invented by him and Ferrars while walking on the Trumpington Road, about the year 1865; and, after consulting many classical scholars, he endeavoured† to substitute the word "Phoronomy" for "Kinematics", but the latter was too firmly established to be changed.

His contributions to mathematical journals were mainly published in the *Quarterly Journal* and the *Messenger* and include "Note on Flexible Surfaces",‡ "The Equations of Equilibrium of a Bent Lamina and of an Inextensible Membrane",§ "Metacentre", and "Tension of Flexible Surfaces",‖ and a number of notes on Dynamics and Geometry. The amount which he wrote for publication would doubtless have been much greater had not so much of his time been devoted to his pupils. At the time when he gave up active work he had the manuscripts of textbooks on several subjects almost ready for publication, compiled from notes which he had prepared for pupils, but, feeling doubtful whether they would be appreciated, he destroyed them. The sum total of all the writing which he did for his pupils was prodigious. When he gave up his rooms in St. John's, about 1902, he presented to the writer a volume of Problem Papers. It was a second volume, and entitled *Some of the Problem Papers set to Pupils between 1860 and 1896*. It contains 105 problem papers, with an average of fifteen problems to each paper, and was accompanied by the solutions in manuscript, all neatly arranged and labelled, and forming a pile of papers fourteen inches high. The solutions are always elegant, and, especially in questions on Geometry and Dynamics, alternative solutions are frequently given, and the problems are made to illustrate as many principles as possible. Many of the questions would now be regarded as old-fashioned and altogether unsuitable for use, but they have served their day and answered their purpose. Modern undergraduates spend much less time in working problem papers than did the pupils of Besant, who was a firm believer in the mathematical traditions of the Victorian era, and

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* *Punch*, 2nd April, 1870:

WHAT CAN THEY BE?

Notes on Roulettes and Glissettes. — This sounds rather frisky for the title of a book by a "Lecturer and late Fellow of St. John's College, Cambridge", and we should feel easier as to the future of St. John's College, if we could receive an assurance from the College Authorities that they have examined the work in question and can vouch for its being of the highest respectability.

† *Nature*, 17th March, 1892.

‡ *Quarterly Journal*, Vol. III, p. 68, 1860.

§ *Quarterly Journal*, Vol. IV, p. 12, 1861.

‖ *Quarterly Journal*, Vol. VIII, pp. 35 and 75, 1867.

strenuously opposed the abolition of the order of merit in the Tripos lists; though, when the reform at length came, he withdrew his opposition in view of the weight of opinion in favour of it.

Always busily occupied until his later years, Besant was a man of wide interests. He was a good classical scholar in his youth, and had some artistic gifts and a fondness for sketching. A man of great geniality and good humour, his relations with his pupils were always of the pleasantest nature. Always interesting and interested, he had a fund of amusing anecdote and used to delight, in his latter days, in memories of the mathematicians of the past.

He died on 2nd June, 1917, in his eighty-eighth year.

A. S. R.

EDWARD KINGSLEY WAKEFORD.

EDWARD KINGSLEY WAKEFORD was born at Plymouth, 15 June, 1894, and killed on the Western Front, 16 July, 1916. It is difficult and painful to write of a life so full, so short, and so precious as his was. The tentative sketch of his career and work which follows is necessarily one-sided and inadequate: it is comforting to reflect that a memorial volume is promised, and that there fuller and more personal accounts of him will appear.

He was the son of Edward and Elizabeth Threapland Wakeford. It may be remarked that his mother was a competent mathematician, and especially excelled in solving problems on conic sections. To his grandfather, the Rev. E. Atkins, he was indebted for much inspiring teaching, while his powers of algebraical manipulation and, even more, his ability as a chess player would remind friends of his uncle, Mr. H. E. Atkins. After spending some time in Hong Kong, where his mother died in 1898, Wakeford was brought back to England at the age of six, and lived, except for a short period, with his grandfather at Leicester. As a small boy he showed precocious interest in the properties of numbers and sometimes in things mechanical. Both then and in vacations afterwards his education owed much to his grandfather, as already indicated.

He attended several preparatory schools, then entered Clifton College as a junior in January 1906, and won a scholarship in the Senior School at the age of thirteen. At school he was mainly distinguished for mathematics. Yet he was a good musician—an extraordinary one, it is said; his interest in the subject was analytical and intellectual. It is worth recording that in English "the fortnightly theme was a vexation to him", for he never liked writing out fair copies of even his best mathematical work.

At Clifton he had the good fortune to encounter a young, energetic and enthusiastic geometer in Mr. W. P. Milne, with the result that he left the school in 1912 possessed of an amount of geometrical knowledge which would have been unprecedented probably in a Tripos candidate, and was wholly astounding in a schoolboy.

He entered Trinity College, Cambridge, as a scholar in October 1912, being already known to some in Cambridge as having produced an essay of merit and originality on the Theory of Cubic Surfaces, which contained the germs of part of his future work. He quickly became known as a young geometer of quite uncommon knowledge and power; the freshness, the vigorous restlessness of his intellect were a delight to contemplate. Truly he had his share of the assertiveness natural to, and so often charming in, youth; in him it was backed up by qualities not so common to youth.

Yet perhaps the most remarkable fact about him in his early days was that before he had been in Cambridge two terms he knew personally practically everyone there who was interested in geometry, and took toll of all. Cambridge geometers were never gregarious, and, in fact, it may be contended that the principal defect in pure mathematics, until quite recently, has been the want of effective co-operation between the erudition of riper years and the enthusiasm, the imagination of youth. Had he been spared, Wakeford would obviously have followed the tracks of some members of his own College in helping to remedy that fatal defect.

His first advanced studies, early in 1913, were in the Theory of Curves and the Theory of Invariants; from this time date the three papers in the *Messenger of Mathematics*, and that in our *Proceedings*, to be mentioned later. The brilliant little article on the Double Six was really written in three or four days (Feb. 1913), as the result of a challenge to construct a proof of the main theorem essentially different from the many that had lately appeared in this country. Now also began his interest in the Theory of Canonical Forms, a subject which was not long out of his mind for the rest of his short life. He quickly succeeded in reducing the binary sextic, and, with the confidence so acquired, was readily stimulated to more ambitious flights by the stimulating remarks of Mr. Richmond on the reduction of the tertiary cubic. A course of Lectures on Modern Analytical Theories, bearing on the distribution of prime numbers, made a very deep impression on him about this time, so deep, indeed, that when his leisure came he had considerable difficulty in deciding whether it should be devoted to analysis or geometry.

As a student nearly every branch of pure mathematics appealed to him; on the other hand he took but little interest in the applied side, and an attempt, made at the request of some of his friends, to interest him in the dynamics of rotation was merely a pleasant failure. He listened good-humoredly enough, but obviously thought that both teacher and pupil were wasting their time. In the Mathematical Tripos of 1914 he naturally enough obtained the highest honours. Yet not perhaps in the decisive style one might have expected, for he was anything but a good examination candidate, and doing a three-hour paper was probably as vexatious to him then as the theme had been in his schooldays. The abolition of the order of merit must have been a godsend to him. Obviously he had never spent (not to say wasted) much time in solving these artificial problems in rigid dynamics, hydrostatics, &c., which were formerly such an essential part of the Cambridge discipline.

After the Tripos he hesitated for some time, as stated, in determining whether to attempt geometry or analysis for a Fellowship dissertation. He had, however, been pondering over Canonical Forms for some time, and, more lately, over the problem of finding the relation connecting seven

lines when they lie on the same quartic surface, so his decision to continue these subjects for the present was not surprising. Then the War came. There was no hesitation on his part, perhaps just a little regret at leaving his problems, certainly some chafing, because not having been in the O.T.C. at Cambridge he had a week's delay in getting his commission.

While training he must have spent most of his scanty leisure on mathematics, for late in 1914 he announced to Dr. Baker some beautiful results on the seven-line problem.

He was slightly wounded early in 1916, and soon after coming home was busy again with Canonical Forms. With the help of Dr. Baker he had discovered a paper of Hilbert's which contained the very theorem he had long been in want of—first vaguely, and later quite definitely. This was in March; April found him, full of the most joyous and reverential admiration for the great German master, working away in fearful haste to finish the dissertation.

It is obvious from the MS. (which was finished and sent in for examination) that he was constantly torn between the desire to develop his ideas more fully and the dire necessity of having the paper finished, for his time was short. He returned to the front in June and was killed in July.

There are four papers published in his name, viz:—

I.—A Property of Three Triangles Circumscribing a Conic. *Messenger of Math.*, Vol. 42.

II.—A Canonical Form of the Binary Sextic. *Ib.*, Vol. 43.

III.—An Arrangement of the Positive Integers in the Type ϵ. *Ib.*, Vol. 43.

IV.—The Double Six. These *Proceedings*, Vol. 15.

There is further the dissertation—

V.—General Theorems and Canonical Forms.

And finally a paper which is at present lost—

VI.—The Condition that Seven Lines should Lie on a Quartic Surface (the title being provisional).

I and IV have in common an underlying idea which had a great attraction for Wakeford. If a twisted cubic is projected from a point of itself on to a plane, it becomes a conic γ. A simple argument shows that chords of the cubic which intersect a fixed line g project into lines in the plane that touch a conic δ, the conics δ and γ being so related that triangles can be inscribed in γ whose sides touch δ. There is thus a correlation between lines g in space and conics δ in a plane related in the above poristic fashion to γ. If two lines g, g' intersect, the corresponding conics δ, δ' are inscribed in a common triangle whose corners lie on γ.

In I, Wakeford used this principle to show that if three triangles circumscribe a conic the three conics that contain their vertices have a

common point. Later he learnt that the theorem was not new, and took the opportunity of giving a proof different from the three former ones. In IV he used it to make the fundamental theorem of the double six depend on the well known theorem of Miguel. Though hastily written in his last leisure, IV is a surprising and delicate piece of reasoning. Surprising because of the unexpected relation between two such different theorems, delicate because almost the slightest deviation from the writer's path leads to puzzling errors, a fact of which he was quite conscious.

II marks his first interest in Canonical Forms. Clearly the special problem here treated was in his thoughts to the end, for the penultimate sentence of his dissertation is a query as to the number of similar reductions for a binary quartic of any even order.

On V it is not very easy to comment adequately as yet. A few words on the genesis of the paper must suffice at present. The reduction of a form to a type called canonical generally depends on indirect methods, *i.e.*, the effective linear transformation is not put into evidence. Wakeford attacked the general problem of finding the transformation coefficients from a system of equations, and was thus led to Hilbert's work. The theorem he relies on is a kind of generalization of Noether's on plane curves, stated as follows:—If a homogeneous form ϕ vanishes for all sets of the variables that make $f_1, f_2, \ldots, f_n$ vanish, then a finite power of ϕ can be expressed as the sum of terms each having an f for factor. He was not really concerned with the number of solutions, but with finding criteria to determine whether there are any. Especially he wished to banish or explain away the words "in general" which so often occur in the enunciation of geometrical theorems. He succeeded, anyhow, in stating the problems that arise in a new and more exact form. The first part of the paper is of a very general algebraic character; in the second, which is nearly independent, there occur many beautiful flashes of the geometrical wit in which he was such an artist.

Of VI there exists (thanks to Dr. Baker) a list of the main results. Seven lines will not lie on the same quartic surface unless there is one relation among their positions. Long ago Cayley showed that the relation is of degree four in each set of line-coordinates, but its explicit form is not known. Wakeford attacked the question by synthetic methods, based on a consideration of the six twisted cubics that have six given lines for chords, and arrived at the conclusion that if seven lines lie on a surface of degree four they also lie on a surface of class four. This curious result obviously throws much light on the form of the relation sought.

Wakeford will not be forgotten by those who knew him, especially by those who taught him, and he ought to be remembered, because of the many who have fallen in the War few made so great a sacrifice as he did. He only needed a chance, and he never got it.

J. H. G.

ERRATA.

Mr. F. B. Pidduck writes :—

Prof. W. McF. Orr has kindly pointed out some serious errors of principle in my paper, "The Stability of Rotating Shafts", *Proc. London Math. Soc.*, Ser. 2, Vol. 9, p. 352, which entirely vitiate the conclusions of that paper. It is to be hoped that Prof. Orr will publish his own investigations as opportunity offers.

PAPERS

PUBLISHED IN THE

PROCEEDINGS OF THE LONDON MATHEMATICAL SOCIETY

ADDRESS BY THE RETIRING PRESIDENT

By Sir Joseph Larmor.

[Read November 2nd, 1916.]

In the midst of the universal cataclysm of the War, when all interests are strained towards the national defence, the London Mathematical Society has passed, without notice, its fiftieth year of activity. The first meeting was held at University College, on January 16th, 1865, and heard an address by Prof. de Morgan on the aims and prospects of the Society. The de Morgan medal is a reminder for us of his predominant share in the inauguration of the Society, which he did not survive long to guide. In the early days the publications consisted of a series of pamphlets separately paged, containing single communications ; the names of Sylvester, Cayley, Harley, Tucker, occur as authors in the first year. There followed later brief reports of meetings, along with papers by de Morgan, Sylvester, Crofton, Cayley, H. J. S. Smith, Cotterill, and others. These publications now stand as Vol. I of the first series of the *Proceedings*. With Vol. II, which begins with the Annual General Meeting of November 8th, 1866, the *Proceedings* became crystallized into a form which has persisted substantially, except as regards size of page, to the present time. The Society began operations with 27 original members, nearly all of them members of University College, London ; at the end of the first year the number of members was 69, rising to 94 in November, 1866 ; and the Society had already become representative of British Mathematical Science,

by having on its roll most of the eminent investigators in our subject belonging to Cambridge and Oxford as well as London.

On January 15th, 1866, it was resolved, "That steps be taken to ascertain on behalf of the Society whether and on what terms rooms can be obtained at Burlington House," and on November 8th a report was made that "by the kindness of the Chemical Society in lending their rooms, the Society had been enabled to hold their meetings at Burlington House, where they now meet for the first time." By 1868, most of the British authorities on Pure and Applied Mathematics of that time, who were resident within reach, including de Morgan, Cayley, Sylvester, Hirst, Crofton, H. J. S. Smith, Archibald Smith, Clerk Maxwell, Spottiswoode, S. Roberts, Clifford, Stirling, had been taking active share in the work of the Society by attendance and service on the Council, as well as by the contribution of papers for discussion at the meetings. We must not omit from this list Lord Rayleigh, whose memoirs illuminated our *Proceedings* for many years; who, stimulated by the increasing importance of the Society, became the donor of our most substantial benefaction, which has largely increased our resources for publication ever since the early days. In November 1870, the Society migrated to rooms occupied also by the British Association, in the house of the Royal Asiatic Society, 22 Albemarle Street, where accommodation was found for the library, of which a nucleus had been formed by the books of Sir J. W. Lubbock, the physical astronomer, presented by his son, afterwards Lord Avebury; and there by successive forms of tenancy we have remained until now.

For some years past the library, rendered valuable by accumulation of scientific journals through exchange, and by donations of books, has quite outgrown the accommodation available: and weighty complaints became frequent that, by overcrowding, the books had become, notwithstanding the zeal of successive honorary librarians, almost inaccessible to members of the Society. The problem, thus pressed upon them from many sides, was taken in hand resolutely by the Council during the last Session: and after various plans had been proposed and closely considered, a solution was reached.

It came to the knowledge of the Council that the Royal Astronomical Society would probably be willing to extend hospitality to the Mathematical Society, as regards both place of meeting and general head-quarters, thereby establishing or rather renewing an alliance between British mathematicians and astronomers, whose activities have always interpenetrated with the closest mutual benefit. Following on the confirmation of this plan, subject to the approval of the Office of Works, arrangements have also been made with great cordiality by the authorities of the

Science Museum at South Kensington, whereby our library will be deposited in their scientific library under a scheme which will maintain full use of it by the members of the Society, in surroundings where the cognate scientific literature, and extensive mechanical applications of mathematical principles, will be accessible for study.

We have therefore the pleasure now of holding the first of our meetings under the new conditions, at Burlington House, in very congenial surroundings.

The necessities of the national emergency have mobilized with striking success the industrial resources of science, hitherto neglected too largely in our defensive organizations. A most welcome result is the increased sense that has arisen of the national value of scientific pursuits: but danger is by no means absent that, in the haste to secure the material fruit, the welfare of the tree of knowledge, the pure and fertile source from which it springs, may be neglected or even impaired, and like others of ancient days, as well as recent times, we may succumb to the temptation "propter vitam vivendi perdere causas."

It is our duty here to take into consideration how our own special energies may best be rejuvenated and renewed, so as to become more effective in the enhanced and purified national life which, as we trust, will emerge from our present ordeal. Mathematical knowledge, in all ages the ally of all sustained and exact activities, is now more indispensable than ever, when our material wellbeing depends so much on scientific engineering in its mechanical, electrical, and chemical forms. The highest commendation of any growing department of research is to be able to say that it is approaching the quantitative, the mathematical, form: many sciences, formerly descriptive and classificatory, are even now struggling to assimilate a mathematical method. But if it is just to claim that other sciences, nowadays even the biological, aspire with increasing success to become mathematical, that is, exact, in structure, there is on the other hand a duty enjoined on mathematicians to see to it that the main stream of their discipline is kept accessible—free from specialities and complexities, which, valuable and promising as they may be and usually are, on their own account, to those capable of cultivating them, are yet for the present outside the current of the main advances of human knowledge. The play of human thought knows of no boundaries: it can pursue and clarify itself without limitation into endless mazes. All the more we must be careful, in reclaiming and cultivating our boundless domains of mental evolution, not to lose touch of one another: if a theorist cannot command the attention of his own generation, he is hardly likely to attract the interest or serve the purposes of posterity. The one criterion that is available of the

value of an addition to pure knowledge, is the human mental interest it can excite. We have our very being inside a well-ordered cosmos, intellectual and material, which it is our highest mental pleasure to explore in all directions, and learn to comprehend; and we have a not unsafe guide in trained instinct and sense of fitness and symmetry, industriously applied, to appraise aright the value of each new departure. Knowledge thus cultivated, on a broad basis, for its own sake, so far from obstructing industrial applications, is their profound source. The study of curves, especially the conic sections, by the Greeks, at home and afterwards at Alexandria, is not, as is sometimes asserted, an example of mere useless mental ramifications happening to receive an application in later ages: it was on the direct path of progress, and formed the material, adequate and effective, because not unduly complex or abstract, on which the ideas of the infinitesimal calculus, and may we add the mechanics of Archimedes and Galileo, were gradually matured. And if it became in Newton's hands the weapon for the elucidation of the doctrine of universal gravitation, whereby human science first reached out securely into the illimitable universe, what analyst will deny the preordained fitness of the association?

There was a time, when the annual output of the Mathematical Society was smaller in bulk than it is now, that many of us made a point of taking an interest in all the papers that it published. It would be a great thing if we could get back again towards that state of affairs. At least two of our most distinguished analysts have in my hearing traced the aloofness, and even aridity, of much recent work to the neglect of geometrical ideas, the potent source in the past of mathematical progress and consolidation, and the vehicle for the diffusion of our science. It seems a strange phase of development, when we consider the preponderant graphical, tentative, and practical bent of the national intellect, and remember how much of our most characteristic progress and originality in theoretical physics has been, for the sake of being comprehensively grasped and mastered by the mind, so concisely wrapped up in geometrical imagery, and so freed from analytical technicalities, as to have been even obscure to communities trained in more formal and syllogistic methods.

There is always risk in getting too far from the main currents of our times: there is the danger, not always avoided, that in the fog of ignorance and the lack of interest, we may encourage expansion in artificial and unfruitful and even tedious ramifications, while criticising and suppressing with rigour worthy but immature attempts in the well-explored regions of our science, where improvements are so important and originality is so difficult. The contrast with the difficulty of obtaining publication at all a century ago, except in brief summary, gives ground for reflection.

Of recent years the question must have presented itself to not a few of our authors whether the *Proceedings*, developing in so abstract a direction, are now quite as suitable a place for the publication of mathematical physics as they were in the days when Maxwell and Kelvin, and Rayleigh and Routh, were frequent contributors. Yet the potent source of even the most abstract branches of modern analysis has lain in the seizure and orderly cultivation of the intuitional ideas, largely cast in geometrical mould, that are forged by physical science in the effort to systematize its observations of the uniformities of the rational world around us. To renew our strength for wider flights, we must return frequently to mother earth. The main feature of the technique of physical mathematics is that we are seldom dealing with a completed, and therefore strictly limited, logical complex: it is of its essence that the specification of the problem is fluent and provisional, always ready to take on new features as the discussion opens out. The student of mathematical physics cannot with safety afford to be a specialist: every department of physics is dovetailed into the other departments and progresses by their aid; knowledge must be as far as possible on an intuitive basis, to prevent it from becoming top-heavy, and all the threads must be in hand. For intuition sees, however imperfectly, all round a problem at a single glance; while analysis afterwards consolidates a permanent structure by fitting brick to brick. Even the most abstract of analysts must work at a disadvantage if he has no informed interest in the problems of external nature for which his analysis might be of assistance; and conversely, even the most recondite constructions of pure analysis would be of interest to a wider audience, if they could be expounded in non-technical manner, without the great detail that is sometimes thought to be essential to the necessary degree of precision. Nature is never irrational, but our main intellectual aim is the redemption of our views of her operations from that reproach: it is the freshly detected and systematically traced concatenations of her working that enlarge our stock of ideas, become for us a source of new generalisations in abstract procedure, giving fresh points of view to be developed and to react in their turn. It is sufficient to cite the names of Cauchy and Riemann, not to mention the supreme examples of Lagrange and Gauss, to show that the most brilliant originality in abstract analysis, and habitude in the intuitions of physical science, can go together, to great mutual advantage.

Fortunately there are signs, abundant on both sides, that the repulsion which somehow arose with us in the last decades between the tentative, yet essentially progressive, though concise, prospecting of mathematical physics, and the stern but limited rigours associated with undiluted pure analysis, is now beginning to be recognized as cramping and un-

natural; it may thus melt away in a better mutual understanding, and may one even say mutual interest, to the great advantage of both disciplines. Our analysts have been turning with success, and with a zest of a kind that seems familiar to their more physical colleagues, to semi-empirical methods in the Theory of Numbers: speculative interest has again arisen even in divergent series, such as would have rejoiced the soul of de Morgan, logician though he was: and the time-worn problems of partitions and combinations have been yielding their secrets to the powerful leverage of an apparatus of arrays and lattices, that may remind us of crystallography and even of thermodynamics.

Our Society has lost by death not a few of her veteran members during my two years of office. Notices of the work of Morgan W. Crofton, W. H. H. Hudson, Benjamin Williamson, have already appeared in the *Proceedings*. In Sir James Stirling, Senior Wrangler of 1860, lately Lord Justice of Appeal, we have lost another of the survivors of our early days, whose interest in our science never flagged, whose mathematical training and gifts were the foundation of a legal and judicial eminence not often arising in a generation. In William Esson, Savilian Professor, and John Griffiths, we have lost two Oxford mathematicians long connected with us. Rev. M. M. U. Wilkinson and Rev. J. White had also been members of long standing, while Prof. F. R. Barrell had more recently joined us. Though F. W. Frankland, an early member, had passed out of sight owing to distance of domicile, his combination of mathematical and philosophical interests had not become dormant. The actuarial mathematician, Emory McClintock, a pioneer in the great development of our science in America, was for many years an interested member of our body. I may be permitted to add the name of John Henry Poynting: though his life-work attached him to sister societies, his wide physical outlook, combined with mental exactness and penetration, have made for him an enduring name in mathematical as well as experimental physics.

It is our pride and sad privilege to recall the names of the cultivators of our science who, in response to their country's appeal in time of national peril, have already laid down their lives on her behalf. In E. K. Wakeford, Scholar of Trinity College, Cambridge, not a few of us had recognized a future leader in geometrical science. A colleague more senior and more widely known, S. B. McLaren, Professor of Mathematics at Reading, coming from Australia and taking a high degree at Cambridge, had become a learned and philosophical investigator in the difficult domain of statistical molecular dynamics and the relations of the æther to material systems: the work which formed the basis of the recent award of an Adams Prize may remain, I fear, unpublished, in any finally revised form.

We are entitled also to recall the name of H. G. J. Moseley, who, though he would not have claimed to be a mathematician, had in a brief and brilliant career at Oxford and Manchester contributed fundamentally to the data of the mathematical physics of the future, by revealing the earliest universal and unmistakably quantitative relation in the fascinating domain of the correlations of the chemical elements.

Such heavy sacrifices of colleagues who could so ill be spared we must deeply deplore, but not as if they were made in vain. May we not detect beyond them, and on account of them, the promise of nobler and more disinterested times, when the vast destruction of perishable material resources will be far more than compensated in the remembrance of the heroism of the youth of our generation, and in the gain in moral and intellectual wealth that it will stimulate as an abiding possession?

The world's great age begins anew,
The golden years return,
The Earth doth like a snake renew
Her winter weeds outworn.
* * * * *
A brighter Hellas rears its mountains
From waves serener far:
A new Peneus rolls his fountains
Against the morning star.

THE FOURIER HARMONIC ANALYSIS: ITS PRACTICAL SCOPE, WITH OPTICAL ILLUSTRATION

By Sir JOSEPH LARMOR.

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THE expression of a periodic function, representable as regards a single period by an arbitrary curve or graph, however complex, in terms of trains of simple undulations with a discrete series of periods, multiples of a fundamental period, has been before the minds of analysts since the days of Euler and Lagrange. To the latter a well-known trigonometrical formula is due, which assigns a train of n simple periodic components so as to pass through n specified equidistant points on the graph. The transition from this finite result to the infinite series of periods representing (under the necessary limitations) the graph in all its detail, is in a way parallel to the transition made by Fourier from his periodic series, with its discrete group of periods, to the Fourier integral, with its continuous range of periods and no limitation to periodicity in the graph which it represents.

It was mainly with the view to rendering the crude numerical sequences of data, supplied by observation of nature, amenable to analytical discussion, that Fourier became closely occupied with these expansions. Moreover, the graphs representing the initial states in physical problems, such as that of conduction of heat which he formulated and worked on as a typical example, can and often do exhibit what to the mathematician is violent discontinuity. Fourier believed, and supported his view by fair reasons, that this was no detriment to the practical application of his method of operation by means of expansion in a series or an integral. But it was not so with great contemporary analysts: Lagrange and Laplace, Poisson even much later, were entirely repugnant to the idea that any degree of graphical or statistical irregularity could be made amenable to analytical representation in this way.* And Fourier's early memoir of

* Cf. the historical account in Riemann, Göttingen *Habilitationsschrift* of 1854, published by Dedekind in 1867; *Werke*, pp. 213–225. Cf. also Arago, *Éloges*, i, p. 341.

1807, in which the content of the *Théorie de la Chaleur* was, as he says (cf. *Discours Préliminaire*, final sections), largely covered, remained at the Institute fourteen years before it was published, in a revised form communicated in 1811, under the care, as it happened, of its author who had become Perpetual Secretary. A large part of the merit of Fourier was in his unwavering insistence for years on the utility and practical advantage of his procedure. Traces of the controversy that surrounded the subject are embedded in the later sections — somewhat miscellaneous, like memoranda (*pensées*) hastily thrown together—in his treatise: and M. Gaston Darboux, in the national edition of 1890, has drawn attention to the fact that Fourier's considerations as to the sufficiency of the expansions at all parts of the range of the graph, however complex the data may be locally, did really go to the root of the matter, though they needed to be systematized, and restricted, into a connected argument by Dirichlet (1827) in order to form a basis for the modern, logically compacted, theory.

In its modern condensed and concise form, nothing could be more cogent and interesting than this argument, as a sheer compulsion of the intellect: perhaps hardly anything could be further removed from an understanding of the physical significance of the terms of the Fourier expansions with which it deals. It affords a striking illustration of the principle that analytical identity consists in its ultimate essence simply of rearrangement of the elements of the expressions under comparison: when the formal collocations correspond to the groupings that present themselves in the physical analysis of nature, a branch of Mathematical Physics emerges.

The broad mind of Fourier delighted in general surveys, in preference to minute analysis: the eloquent digressions of the *Théorie de la Chaleur* were probably a greater stimulus and source of inspiration, in their wide sweep, to the masters of Mathematical Physics of the succeeding generations than a more guarded and rigorous analytical exposition of the method could ever have become.

Limitations to Fourier Analysis.—This problem as to the possibility of mathematical representation of an irregular graph, and its limitations where possible, is now far more important for physical science than even Fourier anticipated. Observations, meteorological, statistical, experimental, have been accumulating in vast collections for decades, even for centuries. The problem is insistent, and must have been much in mind with those responsible:—What is to be done with them? Can anything substantial be made of them towards the progress of scientific principles,

which alone count for progress? To what extent are they worth continuing? Does a century's record of the barometer, after the known diurnal and other periodic parts, of astronomical precision of period, have been determined from it and discussed apart, mean anything positive at all, within the range of our powers of analysis, more than a mere congeries of atmospheric instabilities? These and other statistical problems have been too much neglected, to their own practical detriment, by mathematicians. They may present themselves in an inconvenient manner, hampering the freedom in which the analyst is accustomed to roam; but they are of the essence of the pure science, and do not by any means belong to the rough fringe of technical applications.

The question of the utility of meteorological statistics becomes more emphatic, on account of remarkable success in problems at first sight cognate, achieved concurrently with, even prior to, the activity of Fourier. The theory of gravitation, in Newton's hands, had completely solved the mystery of the tides of the ocean. The practical mathematical problem presented by them, viz., the working out of the subject on dynamical foundations, so as to connect the complex mass of available tidal observations with their thoroughly known main cause, was undertaken in detail with great mastery by Laplace. The hydrodynamical theory of the tides is founded on his work, and has gone on improving on the same lines ever since. Here, as elsewhere, only ideally simplified problems are amenable to complete dynamical treatment: so that the results lie rather in the form of general indications and master principles acquired, than of actual problems resolved. But a different line of procedure was concurrently opened up, which has led, in the hands notably of Lord Kelvin and his colleague Sir George Darwin, to full and most remarkable practical control of the problem. In dynamics, as also in wider fields, periodic causes produce effects possessing the same periods. Now the gravitational attractions of the heavenly bodies, which originate the tides, are all made up of components having periodicities of astronomical permanence and precision. The periods are not commensurate; but each of them is resolvable into a fundamental period of simple undulation and its own harmonics. This array of discrete periods ought to stand revealed in unending trains of simple undulations, in the analysis of the tidal graph of each port where a record is maintained. A method of extracting them from the graph, by determining their amplitudes and phases with sufficient accuracy, is all that is required; and their schedule will then express concisely the tidal characteristics of that port for all time. An actual tidal record, predicted for the future and verified for the past, can be rapidly run off from them by aid of an appropriate and sufficiently accurate mechanical curve-tracer.

But naturally, these permanent periodic components of tide, thus brought out, are not the whole of the fluctuations of level at our port. There is a residue, non-periodic in character unless some periodic influence has been overlooked, due mainly to meteorological causes, wind and rain, thus connected with irregular sporadic instabilities in the atmospheric system, that have no element of recurrence, other than transient, involved in them. The contrast here brought into view between true sustained periodicities in a prolonged graph, and a residue, aperiodic and erratic, or containing only transient and imperfect periodic elements such as *seiches* (cf. *infra*), is fundamental.

We have here recalled the essential distinction between graphs which are strictly periodic and those which are not so restricted. The former are decomposable throughout their whole extent, however long, into trains of simple undulations, with their discrete system of frequencies, multiples of a fundamental. Such types may be superposed to form a compound graph with no apparent periodicity at all, but yet expressible for all its length as a discrete system of periodic simple trains. The wave trains of higher frequency in the series are required, to include the more abrupt irregularities and fluctuations that may occur along the periodic range of the graph: they tail off to insignificance only when fluctuations, abrupt or of very short range, which would introduce periods of corresponding shortness, are not prominent or not important; in fact a limited series of the components represents the graph with fluctuations of very short range smoothed out. This for practical purposes expresses the now well-known analytical restriction to the validity of the representation of a periodic function by a Fourier series, viz., that the number of its fluctuations within the period must be limited.

The general type of graph, not thus restricted to periodicity, requires for its expression a continuous range of frequencies instead of a discrete group: and it is not even expressible in this manner unless it involves only a limited number of fluctuations along the entire graph. The usual case is that of a graph representing a disturbance confined to a finite range along its axis, perhaps tailing away in both directions: when it is reduced to a limited number of fluctuations by smoothing out very minute features, it becomes thus expressible.

Formally, a single Fourier integral can represent two local groups of disturbance, separated along their common axis by any interval, however great: for the sum of two Fourier integrals is expressible as another such integral. But this transformation holds only under a complication which destroys at some stage its practical utility. For example, consider the sum

$$\int f(m)\cos(mx+a)\,dm+\int f(m)\cos(mx-ma+a)\,dm,$$

which represents two identical groups separated by distance a. It is equal to

$$\int F(m)\cos(mx+A)\,dm,$$

where $F(m) = 2f(m)\cos\frac{1}{2}ma, \quad A = -\frac{1}{2}ma+a.$

Thus in the resultant Fourier integral both amplitude and phase become more rapidly fluctuating over given range of period $(2\pi/m)$ as a increases: illustrating the manner in which the representation fails, except for a limited total range of graph. Subject to such limitation, the portion of a Fourier integral that is calculated from a given reach of any graph belongs in detail to that reach, and makes no contribution to the expression of the parts beyond it. The Fourier integral is, we may say, adopting a concept of fundamental importance as regards fields of energy in Mathematical Physics, a localized representation, not merely as regards the function or graph as a whole, but also in respect of each differential group of periods which it contains: this *aperçu* seems to be the germ of wide corollaries in the dialectic of function theory. The integral is the limit of the Fourier series when the fundamental period of the latter is extended indefinitely, but *only when* the periodic function is not of sensible value except in the middle parts of the period, remote from the ends: a Fourier series is, so to speak, localised periodically, and the next range where it rises to importance must be far enough away not to produce practical disturbance, if we are to neglect that part and so pass from the series to the integral.

In the theory of the tides we deal with unending pure wave-trains: so it is also in the theory of sound, especially for maintained sources such as organ pipes. Mechanical graphs of sound-waves are the ideal illustration of periodic undulations. In the theory of radiation, on the other hand, we have never to do with a single source, but always with a vast congeries of atoms, each radiating temporarily and independently. The resulting stream of radiation may be made up of discrete frequencies forming nearly simple long-sustained trains of waves, as, for example, that emitted by a gas glowing with its line-spectrum, and so be periodic in an imperfect sense: or its optical analysis may show a continuous range of frequencies as in natural light. In the latter case, the radiation is not expressible, mathematically, in terms of simple unending trains of waves: still less so, practically, if it is analysed with small resolving power; not so even if the resolving power applied were infinite, unless the train of tumultuous radiation that is analysed is of limited length or duration.

A true permanent periodicity may be held to be foreign to the Fourier integral mode of representation, for it would be indicated by an ordinate of the periodogram or curve of amplitudes becoming infinite. At first sight one might be tempted even to say that the function of this analysis is to reduce the residue of a graph, given by observation or experiment, to a mathematical form amenable to analytical operations, after its truly periodic constituents have been removed by some other process.

But there is more than this: the example of a train of radiation has shown that problems presented by sporadic repetition of transient periodic elements may be as important as those involving pure periods. One may illustrate by the movement of a strongly damped light pendulum suspended amid the eddies of a draft of air: or again by the transient *seiches* that are excited in a partially confined bay, such as the Gulf of Lyons where they were shown in the Marseilles tide-gauges, by the irregular disturbance that reaches it from the open sea. Incipient oscillatory motions of such dynamical type might conceivably form part of the general disturbance of the atmosphere with which meteorology deals. If a range long enough to be a fair sample is analysed by the Fourier process, such transient periods, repeating sporadically, will show themselves on the periodogram of amplitude or of energy (cf. p. 23, *infra*). If the analyses of two adjacent ranges give different results, we must conclude either that the ranges are too short to be fair samples, or else that the indications of periods which they separately show are only accidental.

Practical problems such as those just indicated were prominent in the work of the British Meteorological Council about thirty years ago, under the inspiration mainly of Sir George Stokes and Lord Kelvin. Preparations were then made for the mechanical harmonic analysis of long graphs of atmospheric pressure and temperature: but difficulties in running the Kelvin integrating machine seemed to have stopped the project.* More recently the basic principles of the subject have been much illuminated, incidentally, by the writings of Lord Rayleigh. Prof. Schuster has greatly stimulated its practical side by clearing the ground for an organized attack, on the lines of the optical resolution of radiation; and he has carried the calculations through for the available statistics of sunspots and other cosmical phenomena, with remarkable and instructive results, some of which will come under notice. The different and much more simple problem, already referred to, of the separation from a mass of statistics of a definite permanent periodic component whose period is known, is illus-

* Cf. various notes in Stokes's *Math. and Phys. Papers*, Vol. v.

trated classically by the isolation, and separate discussion as regards origin, of the diurnal oscillation in the data of terrestrial magnetism.*

The principles underlying all such discussions, and defining the degree of their validity, are essentially a subject for mathematical development. The writings of Laplace and Gauss on the cognate subject of the distribution of errors and irregularities must ever remain classical, though now they may be not much studied in detail: and modern extensions, by Karl Pearson and other investigators, for application to rougher material, might be more widely known, in abstract at any rate. In de Morgan's and Sylvester's time it was not unusual for mathematicians even to engage professionally, to some extent, in statistical work on the actuarial side. They could perhaps still do much to diffuse economy of arithmetical effort, to guide statistical undertakings, now becoming prominent in many branches of knowledge, to concise and significant expression. For the magnitude of the statistics that present themselves in the meteorological and other domains too readily leads to contentment with laborious and unfruitful routine, inspired possibly in part by exaggerated fear of the consequences of what is known as a break in the continuity of the record.† Progress might be assisted in all such departments, with benefits not all on one side, by continuous and sympathetic scrutiny of the nature of the operations, and the scope and the degree of validity of the conclusions, and by interest in their improvement, on the part of modern analysts, to whom minute discussion of the nature and limitations of functional discontinuity in other aspects has become highly congenial. No apology is required for reviewing the practical side of the subject, in an elementary way, before the Mathematical Society, especially as a variation from the purely critical and negative attitude on such matters that is not unfamiliar to us.

Analysis of a Restricted Range of the Disturbance: its Degree of Precision.—If a graph of length L is under consideration, the usual method of extracting the part that repeats itself periodically in equal lengths λ, is to cut the graph up into segments of this length, and superpose them by numerical addition or else mechanically: if there are enough segments, the resultant divided by the number of segments approximates to that periodic part of the whole which is sought. Analysis by the

* Carried through by A. Schuster, *Phil. Trans.*, 1899, pp. 467–518; cf. also the posthumously published revision of the entire Gaussian representation by J. C. Adams, *Collected Papers*, Vol. II.

† The question deserves to be put explicitly, in the light of this discussion—What detriment is caused by a break in a meteorological or magnetic record, other than the absence of observations for temporary comparison with those of other stations?

Fourier series is then applicable to the part, periodic in length λ, that is thus isolated, and can resolve it into a train of simple undulations and its harmonics. The amplitudes thus obtained are ordinates in the continuous periodogram of the original graph. In the use of an optical grating, which performs precisely this operation of superposition of equal reaches on trains of radiation, the periodic part thrown off in any direction thus includes, as terms in its Fourier series, elements of the spectra of the various orders; and these can be separated in space by a new optical resolution. It is easy to recognize that this double operation for finding the ordinates in the periodogram of amplitudes is equivalent, except for the repetition* of the same periods in the different harmonic groups, to breaking up the Fourier integral process into two stages: by analysing into a series the mean sample of the graph, as regards each particular wave-length, it brings out one aspect of the relation of the Fourier series to the Fourier integral.

This procedure, which is exact for an infinite length of graph of restricted fluctuation, becomes very uncertain except in cases of special simplicity of structure, when the length L that is analysed is not large compared with λ, the component wave-length we are looking for. This point is best elucidated by the converse operation: take a length L of the simple periodic function $A\cos(2\pi x/\lambda)$, the graph vanishing on both sides of that length, and find what the Fourier analysis will make of it. It will give a continuous range of periods with amplitudes falling off from the central wave-length λ, on both sides, according to the simple diffraction law. The ordinate in the curve of amplitudes, at the point whose abscissa is $\lambda+\delta\lambda$, is

$$A\frac{L}{2\pi\lambda}\cdot\sin\left(2\pi\frac{L}{\lambda}\frac{\delta\lambda}{\lambda}\right)\Big/2\pi\frac{L}{\lambda}\frac{\delta\lambda}{\lambda}.$$

The sharp wave-length λ, that would belong to an infinite train, has been blurred into a range of wave-length, whose relative extent $\delta\lambda/\lambda$ is of the order of magnitude λ/L. By the use of a range L of material, we must not expect to isolate a component of wave-length λ more closely than this, unless it stands out clear by itself. The very high peak in the curve of amplitudes that would represent a real permanent periodicity is here flattened down into a band with central ordinate of height $AL/2\pi\lambda$, so that the areas concerned remain about equal. In other words, the degree of purity obtainable in the resolution of a limited range L of material, containing true periodic elements of wave-length around λ, is given by $\delta\lambda/\lambda=\lambda/L$: when pushed within differential ranges narrower

* This repetition, which occurs only when periods are commensurate, can be avoided by taking out a periodicity when found, and analysing the residue for the others.

than this order of magnitude of $\delta\lambda$, the analysis produces only confused results.*

Thus if there were a permanent $11\frac{1}{8}$ year sunspot period, analysis of material for eighty years could only bring it out as some kind of band of simple periods extending substantially about $\frac{3}{5}$ of a year on each side of the maximum. If the $11\frac{1}{8}$ year period is not permanent—and the analysis of Schuster has brought out the unexpected result that it does not show itself in the previous eighty years' records—the physical problem is reduced to the question why, among the transient periodic elements involved, those of this particular period have assumed such prominence for the range of time that shows it.† But the eleven yearly fluctuations also appear, running parallel as regards phase, in many other physical phenomena, and are therefore probably deep-seated in the Sun: this suggests that they are dynamical, and if so they are probably a permanent phenomenon, composed of true periods. Two true periods, differing by some amount less than a year, of nearly equal amplitudes, and recently in concordant phases, would only imperfectly represent the case: alternatively, the amplitude of the simple undulation might be affected by a factor of other than circular type, which has been prominent in recent times. But more prolonged data are apparently needed for closer discussion on these lines. On a view of mere probability alone, it would seem that close periodic repetition seven or eight times is strong evidence of a permanent rather than an accidental cause.‡

As already stated, the ideal course as regards true permanent periodic terms of known astronomical periods is to eliminate them from the given graph, necessarily of limited extent, in some other manner, and apply the Fourier treatment only to the residue. Of a given reach of this it will

* This is only a rough illustrative statement. According to experiments by C. M. Sparrow on spectroscopic resolving power, "On Artificial Infinitely Sharp Doublets," *Astrophys. J.*, Sept. 1916, $\frac{4}{5}\lambda/L$ would more precisely represent the limit of practical resolution, which corresponds to the vanishing of the subsidiary maximum in the periodogram of intensity after it has assumed its limiting form, a point of inflexion: this value is usually employed in what follows.

† The very remarkable periodogram of Schuster (*Phil. Trans.*, 1906, reproduced in Prof. Sampson's tract on *The Sun*, p. 90) shows a nearly precise diffraction curve for the range 1825–1900, indicating an approximately pure period of $11\frac{1}{8}$ years and little else: while for the previous 75 years, for which the data are more uncertain, the main feature is two very pronounced summits at about 9 and 14 years, both broader at the base than a single pure period would give. The further details are also suggestive. [A discussion "On Permanent Periodicity in Sunspots," by N. Yamaga and the writer, on the lines of the views set out in the text, is in preparation.]

‡ A cognate question—given a range of material, and *assuming* the presence of *one* permanent simple period, to find its most probable value—has recently been treated by J. I. Craig, *M. N. R. Astron. Soc.*, 1916.

provide a local representation, in a form amenable to mathematical and dynamical operations; but it will be more widely significant only so far as different reaches give concordant results.

Here the other question arises: is it of much use to attempt a periodogram analysis when the total material on hand is only, say, a dozen times the length of the component periods that are sought? A true permanent period would reveal itself from such restricted range merely as a band of some width, but its summit would be very near the right place: only when it is high and isolated would its shape give presumption of true rather than transient periodicity. In the case of two adjacent sharp periods the representations, thus widened, would overlap; and the problem of their interpretation would be not unlike that of the separation of two adjacent lines by a spectroscope of insufficient resolving power—the two problems would be the same were we not discussing amplitudes instead of energies. Two adjacent true periods of comparable amplitudes would show as a continuous range of periods, or, if they are very near, mainly as one false period half-way between them. It would seem that nothing but analysis of a greater range of the material could determine whether permanent sharp periods there occur, or merely a diffused and local periodicity which will gradually fade and disappear beyond that range.

In practice, subject to exceptions where there are only a few prominent summits, as in the sunspot case *supra*, it would usually be of no avail, with a limited range of material, to analyse more closely than a moderate fraction of the range $\delta\lambda$ which determines the purity as estimated above; and it is of no use to take the amplitude $f(\lambda)$ over that limit of range as other than constant at its mean value. We can, of course, analyse closer and separate Fourier elements of shorter range, and so more nearly approximating as regards their own graphs to permanent wave trains: but we cannot assert that they represent the material in hand any better.

The treatment of known true astronomical periods, where amplitudes and phases alone remain to be determined, is illustrated by the Laplace-Kelvin analysis of the tides at a given port. If there is a permanent periodic component in the graph, say $A\cos\{(m+\kappa)t+\alpha\}$, where κ is small and unknown, it will be brought out when the graph multiplied by $\cos mt$ is integrated by one of the mechanical integrating methods. The main term in the graph of the integral which the machine turns out is $A/2\kappa.\cos(\kappa t+\alpha)$, which reveals both the amplitude and phase that were sought: there is the drawback that when κ is small a long range is required to bring out its period.* But if as in tidal theory the period is

* Cf. Stokes (*Proc. Roy. Soc.*, 1879; *Math. and Phys. Papers*, Vol. v, pp. 52, 53), who

exactly known, then $\kappa = 0$, and the integral is $A/2\pi.\cos\alpha$ multiplied by the time from $t = 0$. A second operation with the same multiplier in different phase, viz., $\cos(mt+\beta)$, completes the data; and the amplitude and phase are determined. And the latter course is practically effective when the period is only approximately known, for $\sin(\kappa t)/\kappa t$ is stationary at the origin.

Natural Radiation.—Ordinary light is a very complex conglomerate of transient elements, that may be each more or less periodic: it is compounded of pulses or short trains of waves, which might conceivably be all of a single type but probably vary in type according to some statistical law. The shortest segment of a ray of light that we can appreciate separately is of immense length, hundreds of miles at least: for the most rapid detector that can be used requires a sensible fraction of a second to produce its record. Thus only additive qualities can be recognized in light at all, such as energy; and also polarization when that type of regularity has been impressed all along the ray by reflection or otherwise. A great addition to the resources of physical analysis of light, one that has made Physical Optics a science, is obtained by splitting a ray into two halves, and sliding one along the other, thus producing effects of "interference": this has regular features in that it occurs between the halves of split constituent pulses, all slid apart through the same distance. The utter and overwhelming impossibility of any sustained and therefore detectable interference between lights that have not a common source is patent.

If the light is made up of component trains of simple waves of length λ, limited in length and irregularly distributed along the ray, interference will not vanish until the two halves into which each of the trains has been split are slid quite apart. It will reveal itself in a succession of light and darkness as the sliding is increased by each wave length of the train, making L/λ alternations in all. In optical interference arrangements these alternations usually show simultaneously, spread out in a band transverse to the superposed rays.*

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suggested this type of procedure for the case of meteorological graphs. The integrating machine does not seem to have been found efficient. Moreover, in the light of this discussion, our comment would be that the true permanent periods can only be the already known astronomical ones, to be tested in succession.

* But if the superposed rays are passed through a spectroscope, the interference will be assisted to a range far beyond this limit. A channelled spectrum will be produced with alternations that become closer as the one ray is slid further along the other: when they have become so close as to disappear they will be restored by use of a spectroscope of higher dispersive

But in thus following out in detail, by way of illustration, light made up of limited sequences of simple waves, we are restricted to amplitudes not energies: for the latter are only additive when the pulses do not permanently overlap. We might combine two overlapping pulses into a single one of more complex type: but that cannot be carried through either. Recourse must be had to Fourier. We can take a finite reach of the complex radiation, longer than the greatest range of interference that we contemplate employing, but at least some hundreds of the wave-lengths with which we are concerned, and analyse it into its Fourier integral. Each element of that integral is one of the set of Fourier pulses, if we may introduce that term, representative of that length of the radiation and of nothing beyond. We can reason about these pulses, which represent disturbances limited in length, on the previous lines. We can then proceed similarly with the next reach of the radiation. If such reaches, long compared with the order of wave-lengths, and unpolarized, give the same distribution of energy along the spectrum, they satisfy the only applicable test of identity. A reach of radiation is practically homogeneous when its parts thus give identical distributions of energy along their spectra. A reach could be homogeneous to sense, where hundreds or thousands of miles are the least of it that can be observed, which might be very far from homogeneous on a finer scale. In this way we arrive at an answer to the question as to the nature and limits of the irregularity inherent in natural radiation. It might even be constituted, so far as we can analyse it, of the irregular repetition of a single definite type of pulse: to determine this type (usually of few undulations and fading asymptotically) from the law of distribution of the energy along the spectrum, is a definite problem, which could be readily formulated in an integral equation if the pulses did not overlap.*

Energy Curve of Natural Radiation: Asymptotic Features.—It is instructive for practical issues to inquire what would be the form of the

power. The reason is that the instrument resolves the pulses into their Fourier constituent groups, which are each dispersed into shorter range as regards frequency and therefore into greater length of undulation, as the power is increased. Cf. p. 33 *infra*.

* See for an exposition of the general subject on different lines, giving some solved special examples, Rayleigh, *Phil. Mag.*, Vol. 27, 1889, pp. 460–9, Vol. 5, 1903, pp. 238–43: *Scientific Papers*, Vol. III, p. 275, Vol. V, p. 98.

On the general discussion cf. "On the Constitution of Natural Radiation," *Phil. Mag.*, Vol. 10, 1905, p. 574, and Vol. 11, 1906. Also especially papers by Rayleigh, *Phil. Mag.*, Vol. 10, 1905, p. 401, Vol. 11, 1906, p. 123: *Scientific Papers*, Vol. V, pp. 272–82: and subsequent papers. Cf. also p. 39 *infra*.

energy curve of the radiation if the constituent pulses were perfectly irregular, and it were thus devoid of any kind of statistical uniformity for however short a range of its length. Such complete absence of correlation along the ray implies that each linear element of the graph of the total radiation has no relation to any other element. Now for a single element of disturbance situated at the origin, represented by $\phi(x)$, the Fourier expansion, so far as regards wave-lengths much longer than the element itself, is independent of all details of its form, being

$$\frac{A}{\pi}\int \cos mx\, dm, \quad \text{where} \quad A = \int \phi(x)\, dx,$$

so that A is the total area of the graph of the disturbance. Its law of energy, as given by integrated square of disturbance (cf. p. 24 *infra*) is

$$E = \frac{A^2}{\pi}\int dm = 2A^2 \int \lambda^{-2} d\lambda.$$

The energy is thus distributed according to the law $\lambda^{-2}\delta\lambda$, as regards those wave-lengths which are of a higher order of length than the element; but the main part of it remains undetermined in lower wave-lengths.

We can infer only that the curve of energy of wholly uncorrelated disturbance tails off as $2A^2\lambda^{-2}\delta\lambda$. But the facts for natural radiation correspond to a law of type $\lambda^{-4}\delta\lambda$. This means that A must vanish, in other words that as regards area of its graph, the constituent pulse is as much negative as positive.

That property does, in fact, hold good for every mechanically propagated transverse disturbance $\phi(x-ct)$. For, if $\phi(x)$ denote displacement, the disturbance $\psi(x)$ is equal to $d\phi/dt$, which is $-c\,d\phi/dx$, so that $\int \psi(x)\, dx$ is $-c(\phi_2-\phi_1)$, and so is zero for the whole range from end to end.

It is also true for an ordinary electrically propagated pulse for a similar reason, by virtue of the Maxwellian equations: the exception is an electron suddenly projected, which sends out a positive spherical pulse into the æther, the compensating negative one being sent out far behind when the electron is suddenly stopped, and the steady magnetic field of the moving charge, which is left behind it by the initial pulse, is thus abolished. The energy of the spectrum of the impulsive (non-vibratory) part of the Roentgen radiation, as distinct from the sustained wave trains in its spectrum characteristic of different sources, should thus tail off according to the law $\lambda^{-2}\delta\lambda$ as λ increases.

But for ordinary radiation the constituent pulses have a distribution

of disturbance which integrated over their length is always zero. Now the Fourier expression for the limited graph $c\phi'(x)$ is

$$c\int X \sin mx\, dm,$$

where $\pi X = \int \phi'(x) \sin mx\, dx = |\phi(x) \sin mx| - m\int \phi(x) \cos mx\, dx.$

This is equal to $-mA$ for constituents of undulation very long compared with the length of the graph, where

$$A = \int \phi(x) dx.$$

Thus the vibration in the pulse is $-c\pi^{-1}A\int m \sin mx\, dm$; and its energy is

$$E = c^2\pi^{-1}A^2\int m^2 dm = 8\pi^2c^2A^2\int \lambda^{-4} d\lambda.$$

If the pulses do not overlap, the radiation of long period that passes per second is expressed by

$$8\pi^2c^3B\int \lambda^{-4} d\lambda,$$

where $B = \Sigma A^2$ summed for all the pulses per unit length.*

If ideas are fixed on transverse pulses travelling along a stretched cord of unlimited length, this formula will give the distribution of energy of very long wave-lengths, when the linear density of the cord is unity and A is the area enclosed between the pulse-form on the cord and the axis. The total energy of the pulse is

$$2.\tfrac{1}{2}\int \left(\frac{dy}{dt}\right)^2 dx = c^2\int \left(\frac{dy}{dx}\right)^2 dx = -c^2\int y\,\frac{d^2y}{dx^2}\,dx,$$

while

$$A = \int y\, dx.$$

The pulse or wave-group in this case carries momentum equal to half the energy divided by c.†

If these pulses do overlap permanently, there is no resource but to take the shortest sample length of the radiation as a compound pulse: then A is the (residual) area of the undulatory graph which represents each Fourier element of it, and the same conclusion as regards components of very long period will be maintained.

In illustration, now extended to the complete spectrum instead of its long period constituents, radiation consisting of pulses having displacement of type $\phi(x) = be^{-kx}\cos ax$ on the positive side of the origin, with its image on the negative side, thus of Roentgen type, may be considered.

* Cf. the cognate argument of E. T. Whittaker, *M. N. R. Astron. Soc.*, Nov. 1906, p. 87.

† See *Proc. International Math. Congress*, Cambridge, 1912, Vol. I, pp. 197–216, "On the Dynamics of Radiation."

The velocity in one of them analyses into $\int M \cos mx\, dm$, and its total energy is E, where

$$M = \frac{c}{\pi} bm \left\{ -\frac{2}{m} + \frac{m+a}{(m+a)^2+k^2} + \frac{m-a}{(m-a)^2+k^2} \right\},$$

$$E = c^2 b^2 [k^{-1} + k(a^2+k^2)^{-1}].$$

Thus $\pi M^2/E$ is expressed by a formula of type

$$kF\left(\frac{m}{k}, \frac{a}{k}\right);$$

and the distribution of energy in the spectrum of the radiation is represented exactly by $\int \pi M^2 dm$, which is

$$\int dm\, \Sigma \left\{ kE_{a,k} F\left(\frac{m}{k}, \frac{a}{k}\right) \right\},$$

where $E_{a,k}$ is the total energy of one of the constituent groups. This may be elucidated graphically for the various simple cases that are contained in it: for the long component periods, m small, F is

$$\frac{4}{\pi}\left(1+\frac{k^2}{a^2+k^2}\right)^{-1}.$$

We may recall that Lord Rayleigh, and Mr. Jeans, and others following them, have shown that the distribution of energy equipartitioned among the free vibratory modes of an enclosure follows the law of inverse fourth power of the wave-length, with a coefficient which agrees experimentally with the value for natural radiation. The full explanation of this very remarkable agreement, when it comes, will cover the circumstance that it holds good only for the high periods of the enclosure, whereas it is the low periods of natural radiation that are represented by the corresponding formula.

The Fourier Wave-Group.—We have verified that when a graph along t, of homogeneous type, is analysed into a Fourier integral of form

$$\int dm\, f(m) \cos(mt+\alpha),$$

the purity attainable in the resolution depends on the length L of the graph that is employed in the process. Adjacent values of m situated within the limits given roughly by

$$\frac{\Delta\lambda}{\lambda} = \frac{4}{5}\frac{\lambda}{L}, \quad \text{or} \quad \Delta m = \frac{8}{5}\frac{\pi c}{L},$$

will be confused in the result. The effective element of the analysis of the graph will thus not be an unlimited train of oscillations of type

$$\Delta m\, f(m) \cos(mt+\alpha),$$

but will be a limited Fourier group expressible by

$$\int_{m_0-\frac{1}{2}\Delta m}^{m_0+\frac{1}{2}\Delta m} dm\, f(m) \cos(mt+a),$$

which is, when $f(m)$ and a are treated within its range as constants, involving the limitation (cf. *supra*, p. 12) to a smooth distribution of periods,

$$\Delta m\, f(m) \frac{\sin \frac{1}{2}\Delta m t}{\frac{1}{2}\Delta m t} \cos(mt+a),$$

where m now represents m_0, the central frequency for the group.

Writing t', equal to $t-x/c$, for t, the practical element of radiation is thus expressed as a train of travelling waves of constant period, with amplitude falling off on each side of its centre according to the simple diffraction law of type $\sin\theta/\theta$, where $\theta = \frac{1}{2}\Delta m t'$, and so becoming negligible after a distance of the order $2\pi c/\Delta m$ on each side. We have taken for Δm the value $\frac{8}{3}\pi c/L$, as roughly but sufficiently representing the circumstances; thus this gives a range of extent on the two sides about $\frac{3}{4}L$. The effective lengths of the practical elements of the integral, viz., of the Fourier groups into which it resolves the graph, are thus only slightly greater than the length of the original graph itself. No practical question can arise of resolving a given short length of the radiation into infinitely long trains of waves, a paradoxical mode of statement of the Fourier theory which seems to have puzzled even very high authorities.* To analyse sodium light within 10^{-2} of an angstrom unit (10^{-10} metres), at least three quarters of a million wave-lengths are needed (cf. p. 29), extending over about three decimetres of length. Prolonged radiation of white light can be represented in no other way than as an average of some kind of pulses or limited trains; these may be specified most simply as combined into Fourier groups, travelling on along the ray. This specification by discrete groups is the practical aspect of the limitation, that a graph cannot be expressible as a Fourier integral, unless its length is so short that the number of alternations in it that are important for the purpose in view is finite.

This choice of a Fourier group as the practical element of radiation is convenient because it retains its form unchanged after passage across dispersing media, provided the length of path is not excessive, as will appear immediately.

Energy of Travelling Group: Dispersive Medium.—An expression for the energy of a Fourier wave-group will naturally be required. The dis-

* Thus Poincaré (*C. R.*, 120, 1895, p. 757) referring to the idea that a temporary disturbance can be represented by harmonic components extending over all time before and after, merely infers (in a critique of the views of Gouy) "la formule est donc fausse." Cf. also A. Lande, "Ueber ein paradoxon der Optik," *Phys. Zeits.*, June 1915, p. 204.

placement in the medium is

$$\phi(x) = \int dm\, f(m) \cos(mt - nx), \text{ integrated over a range } \pm \tfrac{1}{2}\Delta m.$$

If $$m = m_0 + w, \quad n = n_0 + \left(\frac{dn}{dm}\right)_0 w + \ldots,$$

we have, for this medium, supposed for generality to be dispersive so that n is a function of m,

$$\phi(x) = \int_{-\frac{1}{2}\Delta m}^{\frac{1}{2}\Delta m} dw \left\{ f(m_0) + w f'(m_0) + \ldots \right\} \times \cos\left\{ m_0 t - n_0 x + w\left(t - \frac{dn}{dm}x\right) - \tfrac{1}{2}w^2 \frac{d^2 n}{dm^2} x + \ldots \right\}$$

$$= \Delta m\, f(m_0) \frac{\sin \tfrac{1}{2}\Delta m \left(t - \frac{x}{c'}\right)}{\tfrac{1}{2}\Delta m \left(t - \frac{x}{c'}\right)} \cos(m_0 t - n_0 x),$$

approximately, where $$c' = \left(\frac{dm}{dn}\right)_0.$$

As thus integrated, the individual waves in the Fourier group travel with velocity c equal to $(m/n)_0$; but their amplitudes as distinct from phases, represented by the first factor, travel with the different velocity c', equal to $(dm/dn)_0$. The latter is the velocity of the disturbance itself, and of the energy it conveys. The group, however, begins to change sensibly after travelling for a time T which makes $c\Delta m^2 T d^2 n/dm^2$ a sensible fraction. If the amplitude varies slowly along the group, the total energy in it, kinetic and potential, is practically the value of $2.\tfrac{1}{2}k\int(\text{amp.})^2\,dx$ at any time, say $t = 0$, where k is an elastic modulus of the medium, such that $k/\rho = c^2$. Thus the energy is

$$k\Delta m^2 f(m_0)^2 \int_{-\infty}^{\infty} \left(\frac{\sin px}{px}\right)^2 dx,$$

where $$p = \tfrac{1}{2}\frac{\Delta m}{c'}.$$

In this expression the integral factor is equal to

$$\left| -\frac{\sin^2 px}{p^2 x} \right| + \int \frac{\sin 2px}{px}\, dx,$$

which is $$\frac{\pi}{p} \quad \text{or} \quad \frac{2\pi c'}{\Delta m}.$$

The total energy of the group, half potential and half kinetic, is therefore

$$2k\pi c' f(m_0)^2 \Delta m.$$

As this is proportional to the range Δm of the group, the energies of groups having a common centre are additive even when they are within the same infinitesimal range of frequency.

In so far as groups of finitely different mean periods recover their identity after running through one another, they, of course, maintain their energies too. But while they are partly superposed, before they have run clear of each other, their energy must also be conserved. This requires that during their encounter, even in a non-dispersive medium, there is slight change of form, the aggregate disturbance not being the sum of the travelling individual groups: for the energy of the sum of two groups is the sum of their energies *plus* a mutual product term, which has to be got rid of.

Interference as regards energy between overlapping Fourier groups becomes observable only when they have the same mean period. The estimation of the mutual part of the energy of two such groups $f(m)$, $F(m)$ with their centres at distance b, involves going back to the structure of the group.

Transformation of Energy Integrals.—This we can do by reverting to the more analytical type of procedure by which the distribution of the energy among the constituent groups was first determined by Lord Rayleigh. We take an arbitrary disturbance $\psi(t')$, which may for convenience be conceived as a travelling one emitted from a stationary vibrating source, so that $t' = t - x/c$; it is restricted as regards range and number of fluctuations so as to be of the type expressible in terms of two continuous distributions of Fourier groups, of the *two standard types*, viz., symmetrical and anti-symmetrical with regard to their central points, in the form

$$\psi(t') = \int f_1(m) \cos mt'\, dt' + \int f_2(m) \sin mt' dt'.$$

The total energy is proportional to

$$\int \psi(t')^2\, dx \quad \text{or} \quad c \int \psi(t')^2\, dt',$$

and is given by the formula

$$\int \psi(t')^2\, dt' = \pi \int f_1(m)^2\, dm + \pi \int f_2(m)^2\, dm.$$

This holds whatever be the form of ψ; therefore it must represent the

distribution of the energy among the Fourier groups as well as its total amount.

To prove this proposition: first consider the square of the cosine integral in the expression for $\psi(t')$, say $\psi_1(t')$, in the form

$$\psi_1(t')^2 = \int dm' f_1(m') \int dm\, f_1(m) \cos mt \cos m't.$$

Breaking up the time factor into a sum of cosines, we have, between any assigned limits of t',

$$\int \psi_1(t')^2\, dt' = \int dm' f_1(m) \int dm\, f_1(m)\, \tfrac{1}{2} \left| \frac{\sin(m'-m)t'}{m'-m} + \frac{\sin(m'+m)t'}{m'+m} \right|.$$

As the fluctuation of the other parts is very rapid, the elements of the integral that contribute sensibly to the result are those for which $m'-m$ is very small. This suggests transformation to a new variable $m'-m$; or slightly better on account of symmetry, to the pair of variables

$$(\mu, \nu) = (m'-m,\ m'+m).$$

Then
$$d\mu\, d\nu = 2\, dm'\, dm;$$

and the effective part of the integral is a narrow band along the axis of μ, which is the median line $m' = m$ in the original plane of integration* with respect to coordinates (m', m). Thus within limits $-t'$ to $+t'$ for t', which can be extended indefinitely,

$$\int \psi_1(t')^2\, dt' = \tfrac{1}{2} \int f_1(\tfrac{1}{2}\nu)^2\, d\nu \int d\mu\, \frac{\sin \mu t'}{\mu} = \pi \int f_1(m)^2\, dm,$$

independent of the range of integration. A similar treatment applies to the sine part $\psi_2(t')$, of $\psi(t')$.

The part involving products of the two integrands in $\psi(t')$ is

$$\int dt' \int dm' f_1(m') \int dm\, f_2(m) (\tfrac{1}{2} \sin \mu t' + \tfrac{1}{2} \sin \nu t');$$

it vanishes, the integrand being of opposite signs on the two sides of the origin.

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* The procedure of Rayleigh (*Phil. Mag.*, Vol. 27, 1899; *Scientific Papers*, Vol. III, p. 273) was, following one of Stokes in a cognate analysis (which he had indeed previously noted, cf. Vol. III, pp. 82, 86, 97, as unnecessary, on grounds akin to the present), to introduce a slight attenuation factor $e^{\mp \alpha t'}$ so as to make the integrals thoroughly convergent, and then change the order of integration. It is instructive to verify that a different factor $e^{-\alpha^2 t'^2}$ leads to the same value for the aggregate, but by a very different analysis, indicating different distribution of the elements of energy. The procedure above exhibits how it is that the energy of trains of waves that at first sight purport to be of infinite length, is really finite and local.

The theorem thus established is applicable only to the analysis of a limited range of a fluctuating graph, with zero values beyond the ends of that range: for the total number of fluctuations must in general remain finite. The test of the statistical homogeneity of the graph will then be that different ranges of it give energy periodograms with ordinates in the same ratio and proportional to the range that is analysed.

The superposition of two groups, identical or of common type, but one with its centre shifted a distance b along the axis from that of the other, produces "interference." The total energy is the sum of the separate energies, for the product term fluctuates along the axis, and represents only the succession of interferences. Two interfering groups of the first Fourier type aggregate, writing t' for $t-x/c$, to

$$\psi(t') = \int f_1(m) \cos mt'\, dm + \int f_2(m) \cos\left(mt' - m\frac{b}{c}\right) dm.$$

Thus the product terms in $\int \psi(t')^2 dt'$ integrated over the range from τ_1 to τ_2 are

$$\int_{\tau_1}^{\tau_2} dt' \int dm' f_2(m') \int dm f_1(m) \cos mt' \cos\left(m't' - m'\frac{b}{c}\right)$$

$$= \int dm f_1(m) f_2(m) \int \tfrac{1}{2} \left| \frac{\cos(\mu t' + m' b/c)}{\mu} \right|_{\tau_1}^{\tau_2} d\mu$$

on neglecting terms of very rapid fluctuation,

$$= \int dm f_1(m) f_2(m) \int \sin\left(\mu \frac{\tau_1+\tau_2}{2} + m\frac{b}{c}\right) \frac{\sin \frac{1}{2}\mu(\tau_2-\tau_1)}{\mu} d\mu$$

$$= 2\pi \int \sin m\frac{b}{c} f_1(m) f_2(m)\, dm,$$

when the integration is over the whole range from $-\infty$ to $+\infty$ in τ, as the essential part of the fluctuating integral in μ is the range, then infinitesimal, for which $\frac{1}{2}\mu(\tau_2-\tau_1)$ is of the order of π.* The fluctuation of the local interference, represented by the sine factor under the integral, becomes more rapid as b increases.

When the two adjacent groups are identical, the range of fluctuation is twice the sum of the energies per unit length of the groups taken separately: in fact the total density of energy fluctuates from that of the sum to that of the difference of the disturbances.

* The expression agrees in form with Schuster's result, *Phil. Mag.*, June 1894, p. 533.

*Energies of Groups must be regarded Atomically.**—A train of travelling disturbance has been resolved by analysis limited to a given length of the train, into a system of elementary Fourier wave-groups, of the symmetrical and antisymmetrical types, each of frequency-range Δm, an order of magnitude specified and representing the degree of purity attainable in the analysis; and the energies of such separate groups have been estimated. Each of these groups is itself bounded in length, not abruptly but fading away on both sides: each of them extends beyond the limits of the original disturbance of which it forms a component. The energy of each constituent group is thus made up of parts, some of which lie entirely beyond the bounds of the aggregate disturbance. In what sense can this be a true conception? As regards amplitude and momentum there is no difficulty, because the superposition of elements can cancel out. But for energy this cannot happen, as each element is essentially positive: the group must in fact as regards energy be treated as an indivisible whole.

The energy in the wave-group has been found to be proportional to its range Δm. We can imagine it divided mathematically into sub-groups δm; each of these, being of shorter range of periods, is longer than the original one, so the same circumstance is involved, that each of them possesses energy beyond the sensible boundaries of the original, of which they form an equivalent. But the essential feature is that the aggregate of the energies is unaltered, as $\Sigma\delta m = \Delta m$. We are fully entitled to introduce into our analysis the aggregate energy of each group; for it is of definite amount and is distributable when the group is sub-divided. But in a statistical analysis of fortuitous undulatory disturbance into elementary groups as here defined, it is not allowable to introduce into the discussion the distribution of this energy along the group. The energy must come in atomically, so to speak, by group-aggregates alone; like the energy of a vortex ring in hydrodynamics, or of a travelling electron. It is only under this implied limitation that the fundamental relation in optics acquires a definite meaning, that the energies or intensities of superposed simple trains of radiation are additive: it is true only in aggregates, not in detail: whereas amplitudes and momenta are additive in minute detail along the rays.

Groups Disentangled by Dispersion.—We have recognised that a Fourier pulse advancing through a dispersive medium maintains for a long

* For a treatment of the statistical dynamics of a field of natural radiation on these principles, see *Proc. Roy. Soc.*, 83 A, 1909, pp. 82–95, "On the Statistical and Thermodynamical Relations of Radiant Energy."

time its form, which travels with the group-velocity

$$c' = d\frac{1}{\tau}\Big/d\frac{1}{\lambda},$$

while the individual waves run ahead through this travelling configuration, with the greater velocity $c = \lambda/\tau$, where λ is wave-length and τ is period, —increasing in amplitude as they advance from the rear and then fading away towards the front. A result is that after passing across a prism, if the line of maximum amplitude of wave crests is traced on the wave-pattern, it has become oblique to the wave fronts, or lines of equal phase, which specify the individual crests and are thus now undulating in amplitude. The geometrical wave front or locus of constant phase is in fact determined by constancy of time of passage as regards the individual undulations: the actual disturbance has fallen behind it by a distance $g(c-c')/c$, where g is the thickness of the glass or other dispersive medium that has been traversed, measured along the ray considered. If Δg is the difference of this thickness for the extreme rays of a parallel pencil which as it emerges is of breadth b, the line of maximum of crests makes with the line of phase, or of individual crests, an angle α, usually small in optical applications, such that

$$\tan\alpha = \frac{\Delta g}{b}\left(1-\frac{c'}{c}\right).$$

If a train of prisms is traversed, instead of only one, the right-hand side will be replaced by a summation over all of them: a Fourier pulse incident on the dispersing system with crests uniform in amplitude along the wave fronts, comes out with the maxima of the crests in echelon to the wave fronts at an angle α, as above.

Something not very different from what thus holds for a Fourier group may be anticipated to hold for a pulse consisting of an abruptly limited train of simple waves. For its analysis consists of Fourier pulses all ranging around its own period, which will thus all be treated nearly alike by the dispersive medium; they may be expected to combine again much as before, and thus to emerge as a nearly similar limited train with amplitudes in echelon, of the same depth from front to rear as previously.* But there is a consideration which might seem to place a limit on this anticipation. Suppose the incident pulse is restricted to a single undulation, or even a non-undulatory displacement, as it can be for abrupt Roentgen pulses on the electric theory. We have recognised that the

* Cf. Schuster, *Phil. Mag.*, Vol. 37, 1894, p. 529. The intuitive form into which he has thrown the explanation of the distinctive features of the Talbot bands in a spectrum, is a beautiful illustration of these principles

function of the prism is, not itself to disperse the incident travelling pulse consisting of a finite series of waves, but rather to prepare for spontaneous dispersion by throwing the lines of maximum displacement oblique to the fronts of the waves: the sorting of the disturbance, thus modified, into a fan of ordinary rays will follow spontaneously in the non-dispersive medium after emergence. But if an incident non-oscillatory sharp pulse could emerge as a pulse of the same sharp type, any distinction between phase front and disturbance front would disappear: such an emergent pulse must travel on unchanged at right angles to its front, so that there could be no dispersion. The paradox is avoided on reflecting that the analysis of a sharp non-oscillatory pulse involves prolonged Fourier groups spread over a wide range of periods, whereas the previous case of a finite simple train involved such groups over a very short range.*

These descriptive matters have been put on more precise footing by the answer to the question: What type of pulse or limited train of waves, if any, preserves its form after passage across a dispersive system of prisms or a grating? One answer, the most important but not the only one, is, as already indicated, that the Fourier wave-group, a very small but finite element of the Fourier integral, has this property. For after passage across the prismatic system the train

$$\int dm\, f(m) \cos\left\{ m\left(t - \frac{x}{c'}\right) + \alpha \right\}$$

becomes

$$\int dm\, f(m) \cos\left\{ m\left(t - \frac{x+\xi}{c'}\right) + \alpha \right\},$$

where ξ is the group-retardation, which is proportional after emergence, for each ray to distance y measured up to it along the wave front. The central plane of the original Fourier group was determined as that front for which as origin the phase-angle α vanishes: it is a distance $c'\alpha/m$ in front of the origin to which the pulse has actually been referred in the formula. But in the pulse as emergent the centre is at a distance $c'\alpha/m - \xi$ in front of this origin, where ξ varies along the front on account of difference of path in the dispersing medium. Thus it emerges as a new wave-group, of Fourier type in every respect except that it is sheared so as to make its line of centres oblique to the fronts of the waves by an angle α already determined.

If radiation of given mean period is suddenly set free in a dispersive

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* The dispersive medium might however be such that groups of all periods have the same velocity: then the pulse would emerge as a sharp pulse but of amplitude undulating along it, as Schuster points out: cf. the discussion of these matters in his *Optics*, Ch. XIV.

medium, and travels onwards with front nearly abrupt, the front that is practically detectable will advance with the smaller velocity of its constituent Fourier groups. It is in fact the group that is the tangible entity, but its phase, which regulates interference with other groups, is determined by a velocity different from its own. Thus any method of measuring the velocity of light which operates on definite portions of the light and identifies them during their transmission must, as Lord Rayleigh pointed out, determine the velocity of the constituent groups, when the medium is dispersive. This is true equally* whether with Fizeau we trace the progress of some kind of interruption in the continuity of the light, or with Foucault we turn the light out of its course by repeated reflection from a revolving mirror, so that the altered deflection arising from the rotation can measure the time elapsing between the two incidences. In the latter case the influence of the motion of the mirror—as distinct from its instantaneous position when a constituent group of the radiation meets it—may be shown to be representable by assigning to the mirror a curvature corresponding in practice to a radius hundreds of miles long.

It is on the other hand true that, when one half of a split beam of light is delayed by opposing a film of dispersing substance such as glass or mica in its path, the interval of distance between the resulting bands of interference is determined by the difference of the phases, and these travel with the velocity of the single waves that advance through the group. But even here the prominent feature is the centre of the linear interference pattern, and that is in the position where the centres of the two halves of the Fourier groups, one transmitted through the film of glass and the other travelling alongside it, are coincident. The position of the centre is thus determined by the group-velocity in the dispersive film of glass. This argument by groups of waves is equivalent to the principle of Airy (1833) and Stokes† (1850), applied more recently by Kelvin (1887) to wave-groups such as ship waves on water, that the centre of the interference system is at the place near which the difference of phases of the interfering trains is stationary. Interference will become obliterated when one half of the split wave-group is so much retarded by the glass that it does not any longer sensibly overlap the other half just after emergence; that is (p. 29) when $g(1-c'/c)$ exceeds the length within which the

* For the history of this point, cf. Rayleigh, *Scientific Papers*, Vol. I, p. 537. It may be noticed that for very long paths there will come a time when $\Delta md^2n/dm^2$ on p. 24 *supra* cannot be any longer neglected compared with dn/dm, and the groups will ultimately cease to be permanent.

† Cf. Rayleigh, "Wave Theory," *Ency. Brit.*, 1888; *Scientific Papers*, Vol. III, p. 62. It is now usually quoted as the principle of permanence of phase.

group, as determined by the smallest ranges $\Delta\lambda$ for which wave-lengths are not confused by the apparatus, is of sensible intensity.

Groups at the front of a train cannot possibly advance into a quiescent medium faster than disturbance of their wave-length is transmissible through the medium. But within an established field of undulation, any velocities may occur: in illustration, if two travelling ridges or solitary waves on water cross each other at a small angle, the front will be raised at the place of crossing, and on a flat sea-beach this hump may be observed in shallow water to sweep along the front at surprising speed.

We have noted that a wave-group or finite train of waves, advancing in echelon as regards loci of maximum disturbance, and therefore with amplitude undulatory along each front, will disperse itself spontaneously into regular trains of simple waves even in a non-dispersive medium, each complex ray spreading out into a narrow fan of simple rays of wave-lengths extending over the range corresponding to the Fourier analysis of the train. As the rays of the fan, initially overlapping, begin to disentangle themselves at greater distance, the purity and therefore the depth of the wave-group will increase, tending towards a limit equal to the original number of wave-lengths plus an additional number adjoined by the dispersing apparatus. The latter number is of the same order as the number of lines in the equivalent grating, viz., the one which would produce the same angle of echelon a and the same breadth b of emerging front. These additional undulations are not created by the dispersing apparatus, but are a measure of the degree of definiteness (or purity) with which it reflects the constituents (Δm) of the pulse that are determined by the Fourier mathematical theorem. They are thus evolved from the homogeneity of the disturbance along the ray.

In fact for a grating of N lines $b \tan a$ is equal to $Nn\lambda$ for the n-th order spectrum, while the angle of the diffraction r is connected with the angle of incidence by the relation $\cos r \tan a = \sin r - \sin c$. This is in exact parallel with emergence from a prismatic system; in each case the emergent pulse, before it has become dispersed into the rays of a spectrum, will have fronts with amplitude undulatory along them. Such an undulating front of periodic disturbance, fixed in space, and sustained for a number of periods by the undulations advancing up to it from behind, has itself the properties of the equidistant spaces of a grating, and this is the root of the correspondence of function between prism and grating.* The grating will repeat the group N times, spread along a length $Nn\lambda$ in the n-th order, involving a number of undulations in the wave front that varies as Nn. Thus the prismatic system will add a number comparable

* Cf. Rayleigh, *Phil. Mag.*, Vol. 10, 1905, p. 401; *Scientific Papers*, Vol. v, p. 276.

with Nn to the original number of undulations in the group or pulse, where N is given by

$$Nn\lambda = b \tan \alpha,$$

where

$$\tan \alpha = \frac{\Delta t}{b}\left(1 - \frac{c'}{c}\right) \text{ as above;}$$

so that

$$Nn = \frac{\Delta t}{\lambda}\left(1 - \frac{c'}{c}\right).$$

Influence of Molecular Structure on Transmission.—This limitation of resolving power does not connote any optical imperfection in the material of the prism or grating: it is conditioned solely by breadth of the beam of light. A prism of very small angle would, if infinitely broad, admit of infinite magnification, by microscopic aid, without confusion, of the very short spectrum it produces: there would be no limit to its resolving power. This is illustrated by the objective prisms of small angle used on front of the object glasses of large telescopes: by Lord Rayleigh's principle, it is the thickness of the prism at its base* and not the magnitude of its angle that determines the amount of detail that can be brought out in the spectra of the stars in the field of vision of the telescope to which it is attached.

But a different question arises. Is there any inferior limit to the range $\Delta\lambda$ of an incident Fourier pulse, if it is to pass through the material of the prism without being confused into an aggregate of wider range? It must be the permanence, and constancy of period, of the assisting molecular vibrations in the prism that ensures regular dispersion, just as it is the constancy of the interval between the rulings on a grating: if such uniformity of repetition is restricted say to a succession of N vibrations on an average, one cannot expect purity of propagation within narrower limits than a range $\Delta\lambda$ of order given by $\Delta\lambda/\lambda = 4/5N$.† In the case of a gaseous medium of propagation, whose molecules are in constant motion with velocity of order v, the Fizeau-Doppler effect must come in to limit the definiteness of period of the waves from the excited molecules which

* Determining as above the total distance by which the plane of disturbance is thrown back, owing to its obliquity, from a plane of phase that coincides with it at one side of the beam: the essential circumstances being in fact directly exemplified on an enlarged scale in the Michelson echelon grating. It is at bottom the same thing whether, following Rayleigh's original procedure (cf. *Sci. Papers*, Vol. III, p. 106), we derive the principle from geometrical optics on the basis of the *mathematically* equivalent aggregate of simple wave-trains each with its own velocity depending upon wave-lengths, or as above evoke the direct correspondence between a prism and a grating on the basis of the actual wave-group with its two relevant velocities.

† Thus the analysis, in actual practice, of spectra within 10^{-2} of an Angstrom, pushes N, the number in a sequence of the molecular vibrations, up to the order of a million.

take part in the propagation, roughly to an extent $\Delta\lambda/\lambda = v/c$. For a transmitting gas the same limit to the purity of the radiation is thus imposed by the participation of the sympathetically vibrating molecules, as has been recognised for a radiating gas owing to the varying lengths of the waves from the moving molecular sources. The maximum range of interference determined by Fabry, Perot, and Buisson for a complete spectrum line is substantially in accord with the formula of Lord Rayleigh for the Fizeau-Doppler cause alone. Thus there appears to be no room for the other type of cause, limited lengths of the trains; though its influence would probably be greater than the minimum breadth of line that is observed, if we assumed that each encounter between molecules deranged trains of radiation that were being emitted from them. We must infer, apparently, that either the molecules of the gas are not rotated in their mutual encounters, or else rotation does not affect the orientation of their field of radiation. This view is enforced further by Lord Rayleigh's remark that if the molecular energy were distributed equally between translatory and rotatory modes, the latter should introduce a defect of purity much in excess of the translatory effect.* The circumstance comes to mind in connection herewith, that the constitution of the æther may in a sense inhibit rotational displacement, if for example the magnetic force were taken to represent in some way a velocity. The sharpness of the Zeeman subdivision of spectral lines seems to demand an explanation of the same kind, that the molecule must somehow be isotropic as regards vibration; anyhow it must persist unchanged when the molecule rotates so to speak beneath it.†

Molecular Scattering of Radiation.—When a Fourier wave-group passes through a material medium, the molecular vibrators receive no sudden shock, for the amplitude of the disturbance rises gradually along the group: their sympathetic vibration is gently established in step with the exciting disturbance, and as gently dies away. No free vibrations are excited, such as would arise from abrupt initial circumstances: thus no fluorescence is to be anticipated, unless chemical bonds are snapped, though, when the incident radiation is compound, the elements of it near the molecular periods are selectively scattered. Here the results obtained by R. W. Wood, on scattering and fluorescence from metallic vapours, should form a clue: incident light adjacent to one of the molecular periods (*i.e.*, incident Fourier wave-groups of that period) is found to produce a

* Reference should be made to Rayleigh, "On the Widening of Spectrum Lines" (*Phil. Mag.*, Vol. 29, 1915, pp. 274–284), summing up critically the previous work of himself and others.

† Cf. *Phil. Mag.*, Dec. 1897; *Æther and Matter*, Appendix F.

scattered beam consisting of that period and also a series of others of equidistant frequencies, whose common difference is probably the index of a structural orbital periodicity in the source.

At first sight the concordance of phase of the secondary molecular wavelets emitted along the front, above described, would appear to cancel them out as regards aggregate scattered amplitude, and so vitiate the Rayleigh formula for the intensity of the blue of the sky, from which inferences so striking have recently been drawn. For a random distribution of molecules very closed packed is very nearly a uniform one: and the phases of the wavelets scattered from them are nearly uniformly spaced. But so long as it is random, the most probable value of the total energy of the scattered wavelets is obtained by summation of the energies that would attach to them separately: which means that over any appreciable time the actual scattered energy is thus obtained, by simple addition. This fundamental, in fact crucial, point has been brought out by Lord Rayleigh in various connexions: cf. *Proc. London Math. Soc.*, 1871, in *Sci. Papers*, Vol. I, p. 76; *Ency. Brit.*, 1888, "Wave Theory," § 4, in *Sci. Papers*, Vol. III, p. 52; *Theory of Sound*, Vol. II, ed. 2, § 42*a*. The randomness of phase which is necessary to the validity of the formula for radiation scattered by a gas, would thus *include* and in fact be derived from the random distribution of the molecules and resulting fluctuation in density—though the opposite has recently been urged, notwithstanding discrepancy from observation. Near the critical point of a gas the randomness fails to be adequate on a large scale, on account of local sensitiveness to slight change of pressure and consequent sluggish recovery; and this is indicated by the flickering opalescence that appears. Again in the nearly regular molecular spacing of a crystal the randomness will be only partial, with increased transparency as a result, as has been remarked recently by Lorentz.

Abrupt Pulse in Dispersive Medium.—Even when we contemplate an extremely narrow travelling disturbance, uniform all over its plane, resolution can be made into Fourier groups, each travelling with its own group-velocity when dispersive quality is present. After a short interval of time from the origination of the disturbance, the groups will have separated but slightly; thus they will when combined re-form the initial narrow disturbance, which will so far be propagated with their mean velocity without much spreading. But the groups will in time become separated, which means that the aggregate disturbance will broaden out as it travels. After a long time the groups will have separated practically completely; and the disturbance, originally a sharp pulse, will then pass

a specified point as a succession of simple waves, with lengths gradually changing. Strictly indeed, according to this synthesis, a disturbance would be felt at all distances immediately on its origination, though exceeding slightly: but this can only be because the averaged circumstances represented by the differential equation do not faithfully represent such minute detail.

This analysis would seem to include the fate of a sharp pulse incident directly on the plane face of the dispersive medium from outside. Part of the pulse is transmitted into the medium, where it gradually spreads out as above, as it travels onward. The only question is whether, if the breadth of the pulse is of the order of molecular coarseness of the medium —if for example it belongs to the impulsive part of the Roentgen radiation —the averaged analysis sufficiently represents the initial abrupt stage: if it does not, part of the pulse is dissipated on incidence by shock among the surface molecules, and the rest travels on as here described.* It is not surprising that larger or more massive molecules scatter it more effectively.

But the case would seem to be different for a narrow pulse permanently *maintained* in such manner that it is constrained to travel through the dispersive medium at assigned constant speed. The analogy of the standing surface waves on a stream of water flowing over a submerged obstacle is here in point, the relative motion being the same as if it were the obstacle that travelled in stagnant water. This regular train of free waves will begin to be established as soon as the obstacle is started: the length of the waves will be that corresponding to a wave-velocity that just keeps pace with the obstacle: but their rear boundary will advance with the slower group velocity.† They illustrate forcibly the principle that by virtue of sustained periodic quality in space (grating) or in time (molecular vibrators or elastic periods) a structure can *resolve* an abrupt disturbance which traverses it into its component Fourier trains separated into different directions in space.

This discussion on practical harmonic analysis, rather desultory owing to the circumstances of this troubled time, must now terminate. It will have been observed that the original materials as regards most of the questions that have been submitted to review have been drawn from the writings of

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* This agrees more closely with Lord Rayleigh's view of the Roentgen pulse radiation (*Phil. Mag.*, Vol. 11, 1906, p. 123; *Sci. Papers*, Vol. v, p. 280) than one which I had previously expressed (*Phil. Mag.*, Vol. 10, 1905, p. 574).

† Cf. Lord Rayleigh, *Phil. Mag.*, Vol. 10, 1905, p. 406; *Sci. Papers*, Vol. v, p. 274.

Lord Rayleigh: while as regards the recent practice of the Fourier harmonic analysis, as applied to statistical material, we have been chiefly indebted to the papers of Prof. Schuster. An attempt has been made on revision to indicate these sources by insertion of footnotes: but doubtless there are still omissions.

The tendency of recent abstract analysis on related matters has been to explore the general qualities of the various types of infinite assemblages, rather than to determine the quantitative relations of average or mean which offer themselves for the purposes of physical theory, when the material is too complex for study in detail. The whole subject is so fundamental for modern physical science that a critical survey of what we can expect to achieve in practice, even though it be disputable in parts, may not be unwelcome.

Added March 3rd, 1917.

(*a*) *Effect of Error in Trial Period.*—In extracting the periodic part $F(x)$ of period h from a graph by the process of reduplication (p. 14), which amounts to wrapping it around a cylinder whose circumference is h and taking the mean of the convolutions, the question arises, what will be the effect of an error in estimating the value of the period h? If a trial period $h+\alpha$ is employed instead of the true period h, the mean of the n convolutions is as regards this periodic part $F(x)$, thus equal to $F(x+h)$, one n-th of

$$F(x)+F(x+\alpha)+F(x+2\alpha)+\ldots+F(x+n-1\,.\,\alpha).$$

As α is small, this sum is approximately

$$\frac{1}{\alpha}\int_x^{x+n\alpha} F(x)\,dx.$$

Thus the ordinate at x found from the erroneous value for the period is, when n is large, the mean of the ordinates of the true periodic part $F(x)$ taken over a range $n\alpha$, on the positive side of x when α is positive. Now this true periodic part may be presumed to be a smooth curve: thus if $n\alpha$ is small compared with the period h, the small error α in the trial period will not sensibly affect the form of the periodic part that is derived by means of it. The amplitude of the undulations is thus a maximum for the true period, falling away on each side of it. This principle can receive application, for example, in the discussion of the records of sun spots.

(*b*) The travelling disturbance represented by the integral

$$\int f(m)\cos(mt'+\alpha)\,dm, \quad t' = t - \frac{x}{c},$$

taken within limits of period of narrow range Δm, has been here named (p. 22) a Fourier wave-group. Each such group, when expressed in the form

$$\int f(m)\cos\alpha.\cos mt'.dm - \int f(m)\sin\alpha.\sin mt'.dm,$$

is seen to be constituted of two simple centred groups, one symmetrical on the two sides of the origin of t', the other antisymmetrical. Each of them is limited, and therefore so is every Fourier group, fading away on both sides with increasing distance from the centre—provided the amplitude $f(m)$ and the phase α do not change very substantially within the range Δm. In a dispersive medium, the velocity c depends on m, and the group advances through its undulations with a velocity of its own, but subject to ultimate degradation. However great be the range in m, the group, though in a dispersive medium it is continually changing its form, is always limited in extent, in so far as $f(m)$ and α change only gradually with m: for it is then made up of Fourier unchanging groups for each of which this holds good.

(*c*) It is worth while, pursuing further the remark on p. 12 *supra*, to combine two groups, of the same finite range of period, with centres at distance a apart, into a single Fourier integral. Thus

$$\int f_1(m)\cos(mt'+\alpha_1)\,dm + \int f_2(m)\cos\left(mt' - m\frac{a}{c} + \alpha_2\right)dm$$

is equal to

$$\int F(m)\cos(mt'+\alpha_1+\beta)\,dm,$$

where the amplitude $F(m)$ and phase $\alpha_1+\beta$ are determined by

$$F(m)\cos\beta = f_1(m) + f_2(m)\cos\left(m\frac{a}{c} + \alpha_1 - \alpha_2\right),$$

$$-F(m)\sin\beta = f_2(m)\sin\left(m\frac{a}{c} + \alpha_1 - \alpha_2\right).$$

The "energy" of the total disturbance is expressible by

$$\pi c\int F(m)^2\,dm = \pi c\int f_1(m)^2\,dm + \pi c\int f_2(m)^2\,dm + \pi c\int 2f_1(m)f_2(m)\cos\left(m\frac{a}{c} + \alpha_1 - \alpha_2\right)dm.$$

The total energies of the components are thus additive, except for the last term, which expresses the interference between the two groups. That term diminishes, substantially and without limit, owing to increased fluctuation of the integrand, as a increases.

A similar analysis holds for two wave-groups approaching each other, except that $t-x/c$ and $t+x/c$ are both involved. As they get nearer, the total kinetic energy alters; and if this were all, the waves would have to undergo temporary change of form, to be annulled when they recede after crossing. But, in the case of crossing undulations all propagated with constant velocity, *e.g.* on a cord, it is easy to see that there is a compensating change in the potential energy. Interference of crossing linear waves thus consists, not in gain or loss of total energy, or in redistribution, but in upsets of the balance at each place between the potential and kinetic energies: in the diverging waves of optics it consists usually of redistribution of the total energy over the region that they occupy.

(*d*) Thus when for example two similar limited wave-groups of complex type, at some distance apart, such as would occur in the Fourier integral expressions for two similar functions of limited range, are combined into one Fourier expression, the *mean* curve of the resulting periodogram of amplitude is obtained by summation, independently of what the distance apart of the constituents may be: but this mean curve is affected by large superposed fluctuations which become narrower and closer without limit as the two constituents get further apart. The energies are wholly additive unless the distance apart is small. This progress towards dissolution of the graphical continuity which is essential to physical analysis may be compared with a cognate case: the periodogram representing a finite reach of a graph of radiation (repeated however without limit) consists of equidistant abrupt peaks which, around any assigned wave-length, increase in number and so crowd together as the length of the graph increases, finally merging in a sense with the continuous distribution belonging to the Fourier integral. For a thorough discussion of natural radiant disturbance neither of these transitions to a limit carries the requisite degree of validity: it seems to be necessary to proceed by statistical survey, after the manner of gas-theory, of the actual limited wave-groups that make up the radiation.

(*e*) A periodogram representing an aggregation of disturbances with interrupted periods is made up of superposed constituents, as regards which some general statements can be made. As regards amplitude the constituent belonging to a limited simple wave train has its

maximum ordinate proportional to the length of the train, and its dimensions in breadth inversely proportional: its area is independent of the length of the train, but is more flattened out for short trains. As regards periodograms of the energies of the component trains, which are the important ones because they are additive without reference to phase, the maximum ordinate increases as the square of the length of the train (with breadth narrowing as the length itself) and thus much more rapidly than the energy of the train; while the area is proportional to the length. Long component trains of small total energy give more conspicuous peaks than short trains of greater energy: a narrow peak shooting up in the aggregate periodogram thus implies constituent trains of considerable length, but if our analysis is not close it will appear flattened and broadened. Disturbances devoid of periodicity in their constituent parts give rise to flat types of periodogram, without prominent summits.

(*f*) A Roentgen pulse, however abrupt and narrow, is expressible as a system of Fourier wave-groups, of limited lengths, travelling together. Each of these groups, since it comes on and leaves gradually, is reflected by a crystal in its own proper direction without initial scattering or sudden disturbance: the sharp pulse is thus spread out by the laminar crystalline reflection into the now well known spectrum of regular wave-groups. But this spectrum will be of continuous type, without sharp maxima, if the incident pulse is aperiodic. The selective spectrum of the periodic rays, that arise probably from the volume-vibration of the æther residing in and around the atom of the source, is superposed on this continuous spectrum of the abrupt pulses sent out from the impacts. In the absence of crystalline regularity in the reflector all the waves will be scattered, broken up and lost, inside it.

It is the regularity of spacing that turns back the Roentgen rays: it is the periodicity of the forced molecular vibrations that are excited that refracts and spreads out into a spectrum a pulse of ordinary light.

In both cases each constituent wave-group comes on gradually: so no initial time of preparation is required for the concomitant molecular vibrations to get into regular swing, nor any final time of relaxation, as I had at one time asserted, following Stokes and in opposition to Rayleigh (*Phil. Mag.*, Vol. xi, 1906, p. 123; *Sci. Papers*, Vol. v, p. 279). Fluorescence would have to differ in kind from scattering, being of the nature of storage and transformation in a complex aggregate system, with release conditioned perhaps by structural change.

(*g*) For every finite function that is integrable—and for this type alone the Fourier operations have a meaning—*e.g.* for every function o

"limited total fluctuation," it is now formally established (cf. Hobson, *Real Variable*, § 472) that, whether its Fourier constituents form a series convergent in the necessary minute details or not, their "energies" do always form a convergent sum, and so may claim to represent in detail the energy of the distribution specified by the function analysed. As regards energy then, it would appear that every distribution of disturbance of "integrable" type, without exception, is analysable by the Fourier procedure: and it is thus not surprising that the energy is the one thing with which the corresponding optical analysis of the complex natural radiation can deal.

The mutual "energy" theorem, *supra*, p. 26, in the form appropriate to two Fourier series instead of integrals, is referred back, as regards its precise enunciation in abstract analysis, to Liapounoff at Kharkoff in 1896 (cf. Hobson, *Real Variable*, p. 716). The range of integration is now *limited* (say from $-\pi$ to $+\pi$) and so the theorem obtains more intuitive assent; viz.,

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,\phi(x)\,dx = \tfrac{1}{2}a_0a_0' + \Sigma(a_sa_s' + b_sb_s'),$$

where a_0, a_1, b_1, ... and a_0', a_1', b_1', ..., are the Fourier coefficients for the two functions. The transition to unlimited range and Fourier integrals, under necessary restriction and management,* leads to the formulæ (p. 26) given and utilised by Rayleigh (1889) and Schuster (1894).

(*h*) The remark originated, apparently, long ago, with Sir John Herschel that the process of finding the coefficients in the representation of a function $f(x)$ between limits a and b by a series S_n, of Fourier type with the proper periods, is the same as the process of adjusting to a minimum the aggregate of mean squares of the errors or residues, represented by

$$\int_a^b \{f(x) - S_n\}^2\,dx.$$

It is now formally established, without any limitation on the type of finite "integrable" function that is concerned, that as the number of terms n increases, this minimum mean square always approaches zero: that is, though the residue of the series itself may not become insignificant as regards form and fluctuation, as the number of terms is increased, yet the "energy" belonging to the residue does tend to vanish.

Again, if the summation of the "energies" of the separate terms of a Fourier series is convergent, it is established (Fischer-Riesz)† that the

* Mr. G. H. Hardy has kindly referred me to M. Plancherel, *Rend. Circ. Mat. Palermo* (1910), for the formal justification of this transition for functions of integrable square. On the general topic, cf. also G. H. Hardy, "On the Summability of Fourier's Series," *Proc. London Math. Soc.*, Ser. 2, Vol. 12 (1913), p. 365, and M. Plancherel, *Math. Ann.*, Vol. 76 (1915), p. 315.

† F. Riesz, *Math. Ann.*, Vol. 69 (1910), pp. 449–477.

function represented by the series is "integrable" without limitation, viz., is "completely determined save at an arbitrary set of points of measure zero." Cf. the recent summary by van Vleck, *Bulletin Amer. Math. Soc.*, Vol. XXIII, p. 12. The functions representing any kind of natural disturbance, such as radiation, must thus be "integrable," otherwise in the physical analysis energy will run off into the higher periods without limit. For present purposes these abstract theorems, though probably operative in a wider region of their own, do not seem to extend or modify the accepted physical ideas.

PERIODIC SOLUTIONS OF THE PROBLEM OF THREE BODIES IN THREE DIMENSIONS

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1. *Introduction.*

The purpose of this paper is to treat the problem of three bodies, from the standpoint of periodic solutions, when two of the bodies are spheres of finite mass, and the mass of the third is infinitesimal, and when the finite bodies move in circles and the infinitesimal body moves in an orbit which deviates slightly from the plane of motion of the finite bodies. In the application of the results to the lunar theory the mass of the moon is considered infinitesimal, and the solutions obtained contain all of the terms arising from the moon's mean angular motion, its parallax and the inclination of its orbit to the plane of the ecliptic. Instead of the inclination, however, the rate of regression of the line of nodes is chosen as the independent element. The advantage of this deviation from the method usually followed is that the regression, being a secular variable, can be determined with greater accuracy than the sine of the inclination, which is a linear variable.

The period of synodic revolution of the three bodies and the period of regression of the line of nodes are supposed given, and, since for every particular problem they are given by observation, they are commensurable. The commensurability of these two periods is essential for the discussion given in this paper. When the period of regression is given, the mean period of an oscillation of the infinitesimal body with respect to the plane of motion of the finite bodies can be determined, and it likewise will be commensurable with the synodic period. The period of the solution here obtained is a common multiple of the mean period of oscillation of the infinitesimal body and its period of synodic revolution.

The case where the three bodies move in the same plane was treated by Prof. F. R. Moulton in 1906, and a periodic solution was obtained

whose period is the synodic period of the three bodies.* In his notation the three bodies were considered to be spheres of finite masses E, M, S, but the discussion and results hold for M infinitesimal by putting $M = 0$, and in that case the orbits of E and S reduce to circles. As applied to the lunar theory, the solution contains all terms arising from the mean angular motion of the moon and its parallax. This solution, when M is put equal to zero, is taken as the generating solution of the three dimensional orbit of this paper.

Heretofore, the method of treating problems of this type, when power series expansion was desired, has been to show by means of the Cauchy-Poincaré theorem† the existence of periodic solutions as power series in a parameter which will converge when the modulus of the parameter is sufficiently small, and then to give a method for the formal construction of the solution. But the general case of periodic solutions for a certain class of differential equations has been treated in an existence theorem‡ by Prof. W. D. MacMillan, in which he has shown that the possibility of formal construction of periodic solutions carries with it the proof of convergence of the solutions so obtained. The differential equations of this paper are shown to belong to this class, and the formal construction of a periodic solution is made without the ordinary existence proof, the convergence of the solution thereby being assured. The parameter in which the solution is expanded is a generalisation of the parameter used by Prof. Moulton in the plane, which was the ratio of the synodic period of M to the sidereal period of E and S. The generalisation is determined so that the exponential terms in the solution shall give for the orbit of the infinitesimal body the observed period of regression.

Much use has been made in the discussion of the theory developed in Moulton's "Periodic Orbits." When the three bodies move in a plane the differential equations defining the coefficients of the successive powers of the parameter in the construction of the solution are linear with constant coefficients, and the solutions of them are relatively simple. But for the three dimensional problem the formal construction of the solution consists in solving an infinite number of sets of linear differential equations with periodic coefficients. The general theory for the solution of such sets is given by Moulton and MacMillan in Chapter I§ of the publication already referred to. There is a strong analogy between the entire

* *Trans. of Amer. Math. Soc.*, Vol. 7 (1906), pp. 537–577.

† *Les Méthodes Nouvelles de la Méc. Cél.*, Vol. 1, p. 58.

‡ *Trans. of Amer. Math. Soc.*, Vol. 13 (1911), pp. 146–158.

§ See also Moulton and MacMillan "On the Solutions of certain Types of Linear Differential Equations with Periodic Coefficients," *Amer. Jour. of Math.*, Vol. 33 (1911), pp. 63–96.

treatment of this problem and the treatment of the problem of "Periodic Oscillating Satellites" in Chapter VII,* the analogy being most marked in the formal construction of the solution.

2. *The Differential Equations.*

Let the masses of the finite bodies be E and S. Take the origin at E and let the rectangular coordinates of the infinitesimal body and of S be x, y, z and x', y', z' respectively. Then, the axes being chosen so that $z' \equiv 0$, the differential equations for the motion of the infinitesimal body are

$$(1)\quad \begin{cases} \dfrac{d^2x}{dt^2} + k^2E\dfrac{x}{(r^2+z^2)^{\frac{3}{2}}} = -k^2S\left\{\dfrac{x-x'}{\rho^3} + \dfrac{x'}{r'^3}\right\}, \\ \dfrac{d^2y}{dt^2} + k^2E\dfrac{y}{(r^2+z^2)^{\frac{3}{2}}} = -k^2S\left\{\dfrac{y-y'}{\rho^3} + \dfrac{y'}{r'^3}\right\}, \\ \dfrac{d^2z}{dt^2} + k^2E\dfrac{z}{(r^2+z^2)^{\frac{3}{2}}} = -k^2S\dfrac{z}{\rho^3}, \\ r^2 = x^2+y^2, \quad \rho^2 = (x-x')^2+(y-y')^2+z^2, \\ r'^2 = x'^2+y'^2 = a'^2. \end{cases}$$

Letting n' denote the angular motion of S, we refer the motion to axes rotating about the z-axis with the uniform angular velocity n' by the substitution

$$\begin{aligned} x &= \xi\cos n'(t-t_0) - \eta\sin n'(t-t_0), \\ y &= \xi\sin n'(t-t_0) + \eta\cos n'(t-t_0), \\ x' &= \xi'\cos n'(t-t_0) - \eta'\sin n'(t-t_0), \\ y' &= \xi'\sin n'(t-t_0) + \eta'\cos n'(t-t_0). \end{aligned}$$

Then equations (1) become

$$(2)\quad \begin{cases} \dfrac{d^2\xi}{dt^2} - 2n'\dfrac{d\eta}{dt} - n'^2\xi + k^2E\dfrac{\xi}{(r^2+z^2)^{\frac{3}{2}}} = -k^2S\left\{\dfrac{\xi-\xi'}{\rho^3} + \dfrac{\xi'}{a'^3}\right\}, \\ \dfrac{d^2\eta}{dt^2} + 2n'\dfrac{d\xi}{dt} - n'^2\eta + k^2E\dfrac{\eta}{(r^2+z^2)^{\frac{3}{2}}} = -k^2S\left\{\dfrac{\eta-\eta'}{\rho^3} + \dfrac{\eta'}{a'^3}\right\}, \\ \dfrac{d^2z}{dt^2} + k^2E\dfrac{z}{(r^2+z^2)^{\frac{3}{2}}} = -k^2S\dfrac{z}{\rho^3}, \\ r^2 = \xi^2+\eta^2, \quad \rho^2 = (\xi-\xi')^2+(\eta-\eta')^2+z^2, \quad \xi'^2+\eta'^2 = a'^2. \end{cases}$$

* See also *Mathematische Annalen*, Vol. 73 (1912), pp. 441–479.

The position of the axes may be so chosen at the initial time that the ξ-axis will always pass through S, that is, so that $\xi' \equiv a'$, $\eta' \equiv 0$.

We now pass to semi-polar coordinates by the transformation

$$\xi = r\cos\theta, \qquad \eta = r\sin\theta,$$

and the differential equations become

$$(3)\quad \begin{cases} \dfrac{d^2r}{dt^2} - r\left(\dfrac{d\theta}{dt}+n'\right)^2 + k^2E\dfrac{r}{(r^2+z^2)^{\frac{3}{2}}} = k^2S\dfrac{\partial U}{\partial r}, \\ r^2\dfrac{d^2\theta}{dt^2} + 2r\dfrac{dr}{dt}\left(\dfrac{d\theta}{dt}+n'\right) = k^2S\dfrac{\partial U}{\partial\theta}, \\ \dfrac{d^2z}{dt^2} + k^2E\dfrac{z}{(r^2+z^2)^{\frac{3}{2}}} = k^2S\dfrac{\partial U}{\partial z}, \\ \rho^2 = a'^2 - 2a'r\cos\theta + r^2 + z^2, \\ U = \dfrac{1}{\rho} - \dfrac{r\cos\theta}{a'^2}. \end{cases}$$

Let the mean angular motion of the infinitesimal body, undisturbed by S, be n, and a the semi-axis major of the orbit in which it moves. Then

$$(4)\qquad k^2E = n^2a^3, \qquad k^2(E+S) = n'^2a'^3.$$

Also, to simplify the notation, let

$$(5)\qquad (n-n')(t-t_0) = \tau, \qquad \frac{n'}{n-n'} = m, \qquad \frac{S}{E+S} = \eta.$$

On transforming equations (3) by (4) and (5), they become

$$(6)\quad \begin{cases} \ddot{r} - r(\dot\theta+m)^2 + (1+m)^2\dfrac{a^3r}{(r^2+z^2)^{\frac{3}{2}}} = \eta m^2a'^3\dfrac{\partial U}{\partial r}, \\ r^2\ddot\theta + 2r\dot r(\dot\theta+m) = \eta m^2a'^3\dfrac{\partial U}{\partial\theta}, \\ \ddot z + (1+m)^2\dfrac{a^3z}{(r^2+z^2)^{\frac{3}{2}}} = \eta m^2a'^3\dfrac{\partial U}{\partial z}, \end{cases}$$

where the dots over the variables denote derivatives with respect to τ,

and where

$$(7)\quad \begin{cases} U = U_0+U_2z^2+U_4z^4+\ldots, \\ U_0 = \dfrac{1}{a'}\left\{1+\dfrac{1}{4}\dfrac{r^2}{a'^2}[1+3\cos 2\theta]+\dfrac{1}{8}\dfrac{r^3}{a'^3}[3\cos\theta+5\cos 3\theta]+\ldots\right\}, \\ U_2 = -\dfrac{1}{2a'^3}\left\{1+3\dfrac{r}{a'}\cos\theta+\dfrac{3}{4}\dfrac{r^2}{a'^2}(3+5\cos 2\theta)\right. \\ \qquad\qquad \left.+\dfrac{5}{8}\dfrac{r^3}{a'^3}(9\cos\theta+7\cos 3\theta)+\ldots\right\}, \\ U_4 = \dfrac{1}{8a'^5}\left\{3+15\dfrac{r}{a'}\cos\theta+\dfrac{15}{4}\dfrac{r^2}{a'^2}(5+7\cos 2\theta)+\ldots\right\}, \quad \ldots. \end{cases}$$

The first two equations of (6) are even in z, while the third is odd in z, and the U_{2j} $(j = 0, \ldots, \infty)$ are power series in r/a', whose coefficients are sums of cosines of integral multiples of θ.

For $z = 0$, the first two equations of (6) reduce to the differential equations of motion for an infinitesimal body moving in the same plane as E and S, which was the case treated by Prof. Moulton. Consequently, equations (6) admit the particular periodic solution:*

$$(8)\quad \begin{cases} r = r_0 = a\{1-[\frac{1}{6}+\cos 2\tau]\eta m^2+[\frac{1}{3}+\frac{15}{16}M_1\cos\tau-\frac{7}{6}\cos 2\tau]\eta m^3+\ldots\}, \\ \theta = \theta_0 = \tau+[\frac{11}{8}\sin 2\tau]\eta m^2+[-\frac{15}{8}M_1\sin\tau+\frac{13}{6}\sin 2\tau]\eta m^3+\ldots, \\ z = 0. \end{cases}$$

We now transform (6) by the substitution

$$(9)\qquad r = r_0+p, \qquad \theta = \theta_0+w,$$

and the differential equations become

$$(10)\quad \begin{cases} \ddot{p}+\theta_1^{(c)}(m;\tau)\,p+\theta_2^{(s)}(m;\tau)\,w+\theta_3^{(c)}(m;\tau)\,\dot{w} = X, \\ r_0^2\ddot{w}+\phi_1^{(s)}(m;\tau)\,p+\phi_2^{(c)}(m;\tau)\,\dot{p}+\phi_3^{(c)}(m;\tau)\,w+\phi_4^{(s)}(m;\tau)\,\dot{w} = Y, \\ \ddot{z}+\psi^{(c)}(m;\tau)\,z = Z, \end{cases}$$

where $\theta_1^{(c)}, \ldots, \theta_3^{(c)}, \phi_1^{(s)}, \ldots, \phi_4^{(s)}, \psi^{(c)}$ are power series in m whose coefficients are periodic functions of τ, and the superfixes c and s denote that the co-

* *Loc. cit.*, p. 563.

efficients are cosine and sine series respectively.* Their explicit forms are

$$(11)\quad \begin{cases} \theta_1^{(c)}(m\,;\,\tau) = -\left\{(\dot\theta_0+m)^2+2\dfrac{(1+m)^2\,a^3}{r_0^3}+\eta m^2 a'^3\left(\dfrac{\partial^2 U}{\partial r^2}\right)_0\right\}, \\ \theta_2^{(s)}(m\,;\,\tau) = -\eta m^2 a'^3\left(\dfrac{\partial^2 U}{\partial r\,\partial\theta}\right)_0, \\ \theta_3^{(c)}(m\,;\,\tau) = -2r_0(\dot\theta_0+m), \\ \phi_1^{(s)}(m\,;\,\tau) = 2r_0\ddot\theta_0+2\dot r_0(\dot\theta_0+m)-\eta m^2 a'^3\left(\dfrac{\partial^2 U}{\partial\theta\,\partial r}\right)_0, \\ \phi_2^{(c)}(m\,;\,\tau) = 2r_0(\dot\theta_0+m), \\ \phi_3^{(c)}(m\,;\,\tau) = -\eta m^2 a'^3\left(\dfrac{\partial^2 U}{\partial\theta^2}\right)_0, \\ \phi_4^{(s)}(m\,;\,\tau) = 2r_0\dot r_0, \\ \psi^{(c)}(m;\,\tau) = (1+m)^2\dfrac{a^3}{r_0^3}-\eta m^2 a'^3\left(\dfrac{\partial^2 U}{\partial z^2}\right)_0, \end{cases}$$

where $(\ldots)_0$ means that we put $r = r_0$, $\theta = \theta_0$, $z = 0$ after differentiation.

From (9) we obtain the equations of variation

$$(12)\quad \begin{cases} \ddot p+\theta_1^{(c)}(m\,;\,\tau)\,p+\dot\theta_2^{(s)}(m\,;\,\tau)\,w+\theta_3^{(c)}(m\,;\,\tau)\,\dot w = 0, \\ r_0^2\ddot w+\phi_1^{(s)}(m;\,\tau)\,p+\phi_2^{(c)}(m\,;\,\tau)\,\dot p+\phi_3^{(c)}(m\,;\,\tau)\,w+\phi_4^{(s)}(m\,;\,\tau)\,\dot w = 0, \\ \ddot z+\psi^{(c)}(m\,;\,\tau)\,z = 0. \end{cases}$$

We next generalize the m by putting

$$(13)\qquad m = m_0+\lambda$$

in (10) except where it appears implicitly in τ, and seek to determine a periodic solution for p, w, z as power series in λ, or in some fractional power of λ. The m_0 will be considered fixed throughout the discussion and the λ as a variable parameter, but in the end we must take $\lambda = m-m_0$ in order to have a solution of the physical problem.

With the substitution (13) equations (10) become

$$(14)\quad \begin{cases} \ddot p+\theta_1^{(c)}(m_0\,;\,\tau)\,p+\theta_2^{(s)}(m_0\,;\,\tau)\,w+\theta_3^{(c)}(m_0\,;\,\tau)\,\dot w = P, \\ r_0^2\ddot w+\phi_1^{(s)}(m_0;\,\tau)\,p+\phi_2^{(c)}(m_0;\,\tau)\,\dot p+\phi_3^{(c)}(m_0;\,\tau)\,w+\phi_4^{(s)}(m_0;\,\tau)\,\dot w = Q, \\ \ddot z+\psi^{(c)}(m_0;\,\tau)\,z = R. \end{cases}$$

* This notation to indicate the character of the functions in τ will be used throughout this paper.

It follows from the character of U, r_0, and θ_0 as given in (7) and (8), and from the nature of the transformations (9) and (13), that P, Q, R have the following properties:—

(α) In P and R the coefficients of even and odd powers of w are cosine and sine series respectively of integral multiples of τ.

(β) In Q the coefficients of odd powers of w are cosine series; and the coefficients of even powers of w are sine series, except for two special terms in $\ddot{w}$ and $\dot{p}$ whose coefficients are cosine series.

(γ) In P and Q only even powers of z occur, while R contains only odd powers of z.

(δ) There are no terms in P, Q, or R independent of p, w, z and their first derivatives with respect to τ.

3. *Symmetrical Orbits.*

If the infinitesimal body is passing through one of the nodes at the initial time, and is also at an apse in conjunction with, or in opposition to S, then the initial conditions are

$$(15) \qquad p = a, \quad \dot{p} = 0, \quad w = 0, \quad \dot{w} = \beta, \quad z = 0, \quad \dot{z} = \gamma.$$

It follows from the properties (α), (β), (γ), and the nature of the left members of (14) that, if we change the signs of w, z, and τ and leave the sign of p unchanged, equations (12) are unaltered. It is readily seen that this transformation leaves the initial conditions (15) unaltered. Therefore the solutions also are unaltered, which shows that p is an even function of τ, while w and z are odd functions of τ. That is,

$$p(\tau) = p(-\tau), \quad w(\tau) = -w(-\tau), \quad z(\tau) = -z(-\tau),$$

$$\dot{p}(\tau) = -\dot{p}(-\tau), \quad \dot{w}(\tau) = \dot{w}(-\tau), \quad \dot{z}(\tau) = \dot{z}(-\tau).$$

According to the Cauchy-Poincaré theorem, with the initial conditions (15), equations (14) admit solutions as power series in a, β, γ, λ, converging when the moduli of these parameters are sufficiently small. We could show in the usual way that it is possible to determine the a, β, γ so that the solutions should be periodic, but instead we make use of MacMillan's

theorem* that, *if we are given a set of differential equations*

$$\text{(A)} \qquad \frac{dx_i}{dt} = f_i(x_j, \mu\,;\ t) \quad (i, j = 1, \ldots, n),$$

in which the f_i are analytic in the x_j and μ, vanish for $x_j = \mu = 0$, and have coefficients which are uniform, continuous, and periodic functions of t; and if the f_i converge for $0 \leqslant t \leqslant 2k\pi$, when $|x_j| < \rho_j$, $|\mu| < r$, then the solutions are expansible as power series in μ or some fractional power of μ, converging for $0 \leqslant t \leqslant 2k\pi$, provided $|\mu|$ is sufficiently small. And if the constants of integration can be determined at each step so as to make the solution formally periodic with period $2k\pi$, then the solutions so determined will be periodic and converge for all finite t when $|\mu|$ is sufficiently small.

Now when equations (14) are put into the form of (A), their right members are analytic in p, $\dot{p}$, w, $\dot{w}$, z, $\dot{z}$ and λ, and vanish for

$$p = \dot{p} = \ldots = \lambda = 0\,;$$

and since the cofficients are cosine and sine series of integral multiples of τ, they are uniform, continuous, and periodic. Also the right members converge when the moduli of $p, \ldots, \lambda$ are sufficiently small. Therefore the hypotheses of the theorem are all satisfied, and it follows that if we can determine the constants of integration at the successive steps so as to make the solution formally periodic, we shall be assured of the convergence. It will be shown that it is possible to construct the solution so that it is periodic and symmetrical.

4. *Transformation of the Differential Equations to Facilitate Solution.*

After the generalization (13) has been made, the equations of variation (12) become the left members of (14) set equal to zero; that is,

$$\text{(16)} \quad \begin{cases} \ddot{p} + \theta_1^{(c)}(m_0\,;\,\tau)\,p + \theta_2^{(s)}(m_0\,;\,\tau)\,w + \theta_3^{(c)}(m_0\,;\,\tau)\,\dot{w} = 0, \\ r_0^2\ddot{w} + \phi_1^{(s)}(m_0\,;\,\tau)\,p + \phi_2^{(c)}(m_0\,;\,\tau)\,\dot{p} + \phi_3^{(c)}(m_0\,;\,\tau)\,w + \phi_4^{(s)}(m_0\,;\,\tau)\,\dot{w} = 0, \\ \ddot{z} + \psi^{(c)}(m_0\,;\,\tau)\,z = 0. \end{cases}$$

* *Loc. cit.*, p. 157.

The problem, as already stated, is to obtain a periodic solution of (14) as power series in λ, or in some fractional power of λ, and we further desire to have it symmetric. In constructing the solution, the differential equations defining the coefficients of the successive powers of the parameter will be linear, and, in general, non-homogeneous. The method* of solving at the successive steps is to obtain the general solution of the homogeneous parts set equal to zero, and then to find the solution of the equations with the right members included by the method of variation of parameters. But the homogeneous parts of the differential equations for corresponding successive coefficients will be the same, viz., the left members of (14), and therefore the general solution of the homogeneous parts set equal to zero will be the general solution of the equations of variation.

The solutions of these equations (16) can be obtained by the general theory of linear differential equations with periodic coefficients referred to in the introduction, but the construction of the solution is much simplified if the term in $\dot{w}$ can be eliminated from the second equation; for in that case it has been shown† that the determinant of a fundamental set of solutions is constant, and that the characteristic exponents taken in pairs are equal in value but opposite in sign. Consequently, we make the substitution

$$w = f(\tau)\, u, \tag{17}$$

and equate to zero the coefficient of $\dot{u}$ in the second equation. From the form of $\phi_4^{(s)}(m_0; \tau)$ in (11), this coefficient becomes

$$2r_0\dot{r}_0 f + 2r_0^2 \dot{f};$$

and therefore f must satisfy the differential equation

$$\dot{r}_0 f + r_0 \dot{f} = 0.$$

The solution of this equation is

$$f = \frac{\text{a constant}}{r_0},$$

and for the sake of simplicity the constant may be taken equal to unity.

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* Moulton and MacMillan, *loc. cit.*, p. 91, § 14.

† *Periodic Orbits*, §§ 18, 33.

Then the substitution (17) becomes

$$w = \frac{u}{r_0} . \tag{18}$$

On transforming (14) by (18), the differential equations become

$$\begin{cases} \ddot{p} + \theta_1^{(c)}(m_0\,;\,\tau)\,p + \theta_4^{(s)}(m_0\,;\,\tau)\,u + \theta_5^{(c)}(m_0\,;\,\tau)\dot{u} = P, \\ r_0\ddot{u} + \phi_1^{(s)}(m_0\,;\,\tau)\,p + \phi_2^{(c)}(m_0\,;\,\tau)\,\dot{p} + \phi_5^{(c)}(m_0\,;\,\tau)\,u = Q, \\ \ddot{z} + \psi^{(c)}(m_0\,;\,\tau)\,z = R, \end{cases} \tag{19}$$

where

$$\begin{cases} \theta_4^{(s)}(m_0\,;\,\tau) = -\left\{ \eta m_0^2 a'^3 \frac{1}{r_0} \left(\frac{\partial^2 U}{\partial r\,\partial\theta}\right)_0 - 2\frac{\dot{r}_0}{r_0}(\dot{\theta}_0 + m_0) \right\}, \\ \theta_5^{(c)}(m_0\,;\,\tau) = -2(\dot{\theta}_0 + m_0), \\ \phi_5^{(c)}(m_0\,;\,\tau) = -\left\{ \ddot{r}_0 + \eta m_0^2 a'^3 \frac{1}{r_0} \left(\frac{\partial^2 U}{\partial\theta^2}\right)_0 \right\}. \end{cases} \tag{20}$$

Since r_0 is an even periodic function of τ and (18) is linear and homogeneous, it follows that P, Q and R will still have the properties (α), ..., (δ) when w is replaced by u.

The equations of variation have become

$$\begin{cases} \ddot{p} + \theta_1^{(c)}(m_0\,;\,\tau)\,p + \theta_4^{(s)}(m_0\,;\,\tau)\,u + \theta_5^{(c)}(m_0\,;\,\tau)\dot{u} = 0, \\ r_0\ddot{u} + \phi_1^{(s)}(m_0\,;\,\tau)\,p + \phi_2^{(c)}(m_0\,;\,\tau)\,\dot{p} + \phi_5^{(c)}(m_0\,;\,\tau)\,u = 0, \\ \ddot{z} + \psi^{(c)}(m_0\,;\,\tau)\,z = 0. \end{cases} \tag{21}$$

If we now find periodic solutions of (19) as power series in λ, or in some fractional power of λ, then by the relation (18) we can obtain the desired solutions of (14). That is, our problem has become the solution of (19), and therefore, as shown before, it is necessary to find the general solutions of the equations of variation. The first two of equations (21) are independent of the third and may be integrated separately.

5. *Solutions of the First Two of Equations* (21).

The character of the solution of these two equations depends upon the nature of their characteristic exponents. For $m_0 = 0$, the equations re-

duce to

$$\ddot{p} = 3p + 2\dot{u},$$

$$\ddot{u} = -2\dot{p}.$$

Hence the characteristic equation is

$$h^2(h^2+1) = 0.$$

Therefore two of the characteristic exponents are zero and the other two are $\pm\sqrt{-1}$.

The two solutions corresponding to $h = \pm\sqrt{-1}$ are of the form*

$$(22) \qquad \begin{cases} p = e^{\omega\sqrt{-1}\,\tau}\,\xi^{(1)}(\tau), & u = e^{\omega\sqrt{-1}\,\tau}\,\eta^{(1)}(\tau), \\ p = e^{-\omega\sqrt{-1}\,\tau}\,\xi^{(2)}(\tau), & u = e^{-\omega\sqrt{-1}\,\tau}\,\eta^{(2)}(\tau), \end{cases}$$

where ω is a power series in m_0 with constant coefficients, and $\xi^{(i)}$, $\eta^{(i)}$ are power series in m_0 whose coefficients are periodic with the period 2π. Without loss of generality we may take $\xi^{(1)}(0) = \xi^{(2)}(0) = 1$.

Equations (21) are unchanged by changing the sign of $\sqrt{-1}$, and hence, if we put $\sqrt{-1} = -\sqrt{-1}$ in a solution, the result will be a solution. Suppose, for example, that we change the sign of $\sqrt{-1}$ in the first of solutions (22). Then the initial value of $\xi^{(1)}$ will be unchanged, since it is real, and the characteristic exponent will change sign since it is purely imaginary. That is, the solution will have the initial value and characteristic exponent of the second solution. But since by general theory the solution is unique when the characteristic exponents and the initial value of one of the dependent variables has been given, it follows that

(23)

$$\omega(\sqrt{-1}) = \omega(-\sqrt{-1}),\ \xi^{(1)}(\sqrt{-1}) = \xi^{(2)}(-\sqrt{-1}),\ \eta^{(1)}(\sqrt{-1}) = \eta^{(2)}(-\sqrt{-1}).$$

Therefore the coefficients of ω expanded as a power series in m_0 are all real.

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* The exponents differ only in sign throughout because changing the signs of both u and τ leaves the differential equations and the initial values of the $\xi^{(i)}$ unchanged. Therefore, in virtue of the uniqueness of the solution ensured by the general theory, the two solutions for p in (22) differ only in the sign of τ.

Equations (21) are unchanged if we write

$$p = \bar{p}, \quad u = -\bar{u}, \quad \tau = -\bar{\tau},$$

and the initial conditions are unchanged by this transformation. Therefore

$$\xi^{(1)}(\tau) = \xi^{(2)}(-\tau), \quad \eta^{(1)}(\tau) = -\eta^{(2)}(-\tau). \tag{24}$$

It follows from (23) and (24) that when $\xi^{(i)}$, $\eta^{(i)}$ are expressed as Fourier series they have the form

$$\begin{cases} \xi^{(1)} = \sum_{j=0}^{\infty} (A_j^{(1)} \cos j\tau + \sqrt{-1}\, A_j^{(2)} \sin j\tau), \\ \xi^{(2)} = \sum_{j=0}^{\infty} (A_j^{(1)} \cos j\tau - \sqrt{-1}\, A_j^{(2)} \sin j\tau), \\ \eta^{(1)} = \sum_{=}^{\infty} (\sqrt{-1}\, B_j^{(1)} \cos j\tau + B_j^{(2)} \sin j\tau), \\ \eta^{(2)} = \sum_{j=0}^{\infty} (-\sqrt{-1}\, B_j^{(1)} \cos j\tau + B_j^{(2)} \sin j\tau), \end{cases} \tag{25}$$

where $A_j^{(i)}$, $B_j^{(i)}$ $(i = 1, 2\,;\ j = 0, \ldots, \infty)$

are power series in m_0 with real coefficients.

Two more solutions could be obtained by the general theory from the zero characteristic exponents, but it is possible to obtain, in a simpler manner, two particular solutions which with (21) constitute a fundamental set.

Equations (8) are the *generating solution*, and it is shown in § 32 of "Periodic Orbits" that particular solutions of the *equations of variation* are obtained by partial differentiation of the generating solution with respect to arbitrary constants, if such are present. Two arbitrary constants are present in (8), viz., the initial time t_0 and the scale factor a. Therefore two solutions of equations (12) are

$$\begin{cases} p = \dfrac{\partial r_0}{\partial t_0}, & w = \dfrac{\partial \theta_0}{\partial t_0}, \\ p = \dfrac{\partial r_0}{\partial a}, & w = \dfrac{\partial \theta_0}{\partial a}. \end{cases} \tag{26}$$

According to (5),

$$\tau = (n-n')(t-t_0) = n'/m\,(t-t_0),$$

and therefore

$$\frac{\partial}{\partial t_0} = -\frac{n'}{m}\,\frac{\partial}{\partial \tau}.$$

But since $-n'/m$ is a constant and may be absorbed in the constant of integration later, it follows that partial differentiation with respect to τ will give one solution, viz.,

$$(27)\quad \begin{cases} p = \dfrac{\partial r_0}{\partial \tau} = a\{[2\sin 2\tau]\,\eta m^2 - [\tfrac{15}{16}M_1 \sin\tau - \tfrac{7}{3}\sin 2\tau]\,\eta m^3 + \ldots\}, \\ w = \dfrac{\partial \theta_0}{\partial \tau} = 1 + [\tfrac{11}{4}\cos 2\tau]\,\eta m^2 + [-\tfrac{15}{8}M_1\cos\tau + \tfrac{13}{3}\cos 2\tau]\,\eta m^3 + \ldots. \end{cases}$$

It follows from (4) and (5) that a and m are related by the equation

$$(28)\quad a = \frac{m^{\frac{2}{3}}}{(1+m)^{\frac{2}{3}}}K,$$

where K is a constant which is distinct from zero. Therefore partial differentiation with respect to m, instead of with respect to a, will give the other solution equally well. The m also appears implicitly in τ, so that we have symbolically

$$(29)\quad \frac{\partial}{\partial m} = \left(\frac{\partial}{\partial m}\right) + \frac{\partial a}{\partial m}\,\frac{\partial}{\partial a} + \frac{\partial \tau}{\partial m}\,\frac{\partial}{\partial \tau},$$

where $(\partial/\partial m)$ means partial differentiation with respect to m so far as it appears explicitly. From the equation

$$\tau = n'/m\,(t-t_0),$$

we obtain

$$(30)\quad \frac{\partial \tau}{\partial m} = -\frac{\tau}{m},$$

and from (28) we obtain

$$(31)\quad \frac{1}{a}\,\frac{\partial a}{\partial m} = \frac{2}{3}\,\frac{1}{m(1+m)},$$

On using (29), (30), and (31), another solution of (12) is

$$(32)\quad \begin{cases} p = \left(\dfrac{\partial r_0}{\partial m}\right) + \frac{2}{3}\dfrac{a}{m(1+m)}\dfrac{\partial r_0}{\partial a} - \dfrac{\tau}{m}\dfrac{\partial r_0}{\partial \tau} \\ \quad = a\left[\dfrac{2}{3m}\{1-m+[1-4(\frac{1}{6}+\cos 2\tau)\eta]m^2+\ldots\} \right. \\ \qquad\qquad \left. -\dfrac{\tau}{m}\{[2\sin 2\tau]\eta m^2+\ldots\}\right], \\ w = \left(\dfrac{\partial \theta_0}{\partial m}\right) - \dfrac{\tau}{m}\dfrac{\partial \theta_0}{\partial \tau} \\ \quad = \{[\frac{11}{4}\sin 2\tau]\eta m+\ldots\} - \dfrac{\tau}{m}\{1+[\frac{11}{4}\cos 2\tau]\eta m^2+\ldots\}. \end{cases}$$

Now the only difference between equations (12) and equations (16) is that in the latter m has been replaced by m_0, except where it appears implicitly in τ. Therefore, if we replace m by m_0 in (27) and (32), except where it appears implicitly in τ and where it has arisen from $\partial\tau/\partial m$, the results will be solutions of (16), and may be written in the form

$$(33)\quad \begin{cases} p = a\psi_1^{(s)}(\tau), \quad w = \psi_2^{(c)}(\tau), \\ p = a\{\chi_1^{(c)}(\tau)+\tau\psi_1^{(s)}(\tau)\}, \quad w = \chi_2^{(s)}(\tau)+\tau\psi_2^{(c)}(\tau), \end{cases}$$

where

$$(34)\quad \begin{cases} \psi_1^{(s)} = [2\sin 2\tau]\eta m_0^3 - [\frac{15}{16}M_1\sin\tau - \frac{7}{3}\sin 2\tau]\eta m_0^4+\ldots, \\ \psi_2^{(c)} = m_0+[\frac{11}{4}\cos 2\tau]\eta m_0^3+[-\frac{15}{8}M_1\cos\tau+\frac{13}{3}\cos 2\tau]\eta m_0^4+\ldots, \\ \chi_1^{(c)} = -\frac{2}{3}m\{1-m_0+[1-4(\frac{1}{6}+\cos 2\tau)\eta]m_0^2+\ldots\}, \\ \chi_2^{(s)} = -m\{[\frac{11}{4}\sin 2\tau]\eta m_0^2+[-\frac{45}{8}M_1\sin\tau+\frac{13}{2}\sin 2\tau]\eta m_0^3+\ldots\}. \end{cases}$$

Finally, since $u = r_0 w$, the corresponding solutions of (21) are

$$(35)\quad \begin{cases} p = \psi_1^{(s)}(\tau), \quad u = \psi_3^{(c)}(\tau), \\ p = \chi_1^{(c)}(\tau)+\tau\psi_1^{(s)}(\tau), \quad u = \chi_3^{(s)}(\tau)+\tau\psi_3^{(c)}(\tau), \end{cases}$$

where $$\psi_3^{(c)} = \psi_2^{(c)}\frac{r_0}{a} = \psi_2^{(c)}\{1-[\tfrac{1}{6}+\cos 2\tau]\eta m_0^2+\ldots\},$$

$$\chi_3^{(s)} = \chi_2^{(s)}\frac{r_0}{a} = \chi_2^{(s)}\{1-[\tfrac{1}{6}+\cos 2\tau]\eta m_0^2+\ldots\}.$$

It is obvious from the forms of (22) and (35) that no linear relation can exist among these four solutions, and moreover direct computation shows that their determinant at $\tau = 0$ is different from zero for $m_0 = 0$, and is therefore different from zero for $|m_0|$ sufficiently small. Therefore (22) and (35) form a fundamental set of solutions. Hence the general solution of the first two equations of (21) may be written in the form

(36)

$$\begin{cases} p = a_1 e^{\omega\sqrt{-1}\,\tau}\,\xi^{(1)}(\tau) + a_2 e^{-\omega\sqrt{-1}\,\tau}\,\xi^{(2)}(\tau) + a_3\psi_1^{(s)}(\tau) + a_4\{\chi_1^{(c)}(\tau) + \tau\psi_1^{(s)}(\tau)\}, \\ u = a_1 e^{\omega\sqrt{-1}\,\tau}\,\eta^{(1)}(\tau) + a_2 e^{-\omega\sqrt{-1}\,\tau}\,\eta^{(2)}(\tau) + a_3\psi_3^{(c)}(\tau) + a_4\{\chi_3^{(s)}(\tau) + \tau\psi_3^{(c)}(\tau)\}, \end{cases}$$

where

$a_1, \ldots, a_4$ are arbitrary constants,

$$\xi^{(i)} = \sum_{j=0}^{\infty} \xi_j^{(i)}(\tau)\, m_0^j, \quad \xi_j^{(i)}(\tau + 2\pi) = \xi_j^{(i}(\tau) \quad (i = 1, 2;\ j = 0, \ldots, \infty),$$

$$\eta^{(i)} = \sum_{j=0}^{\infty} \eta_j^{(i)}(\tau)\, m_0^j, \quad \eta_j^{(i)}(\tau + 2\pi) = \eta_j^{(i)}(\tau) \quad (i = 1, 2;\ j = 0, \ldots, \infty),$$

$$\omega = 1 + \sum_{j=1}^{\infty} \omega_j m_0^j,$$

$\xi_0^{(i)}$, $\eta_0^{(i)}$ are constants, $\xi_0^{(1)} = \eta_0^{(1)} = 1$.

6. *Solution of the Third Equation of* (21).

The third equation of (21) belongs to the type of differential equations treated by Prof. Moulton in the *Rend. Circ. Mat. Pal.*, Vol. 32 (1911), pp. 348–364. The discussion was there divided into Cases I, II, and III, according as the characteristic exponents were not integers, integers different from zero, or zero respectively, and a method was given for the direct construction of the solution in each case. When $\psi^{(c)}(m_0\,;\ \tau)$ is expanded as a power series in m_0, we see that the third equation of (21) naturally belongs to Case II, but, since the construction of solutions in Case I is considerably simpler than it is in Case II, we bring the equation under Case I by a generalisation of m_0. On expanding $\psi^{(c)}(m_0;\ \tau)$, the differential equation may be written in the form

$$\ddot{z} + \{c^2 + Z_2^{(c)}(\tau)\, m_0^2 + Z_3^{(c)}(\tau)\, m_0^3 + \ldots\}\, z = 0,$$

where

$$
(39)\quad
\begin{cases}
c^2 = 1+2m_0+\left(1+\frac{3}{2}\eta\right)m_0^2-\frac{9}{32}m_0^4+4m_0^5 \\
\qquad +\left(\frac{34}{3}+\frac{2115}{512}M_1^2\right)m_0^6+\left(15+\frac{4455}{256}M_1^2\right)m_0^7+\ldots, \\
Z_2^{(c)} = 3\eta\cos 2\tau, \\
Z_3^{(c)} = -\left[\frac{45}{16}M_1\cos\tau-\frac{19}{2}\cos 2\tau\right]\eta, \\
Z_4^{(c)} = -15M_1\cos\tau+\frac{40}{3}\cos 2\tau+\frac{75}{64}M_1\cos 3\tau+\frac{33}{8}\cos 4\tau, \\
Z_5^{(c)} = -\frac{18411}{256}M_1\cos\tau+\frac{86}{9}\cos 2\tau-\frac{705}{128}M_1\cos 3\tau+\frac{2937}{32.5}\cos 4\tau, \\
Z_6^{(c)} = -\frac{310325}{1024}M_1\cos\tau+\left(\frac{18709}{256.27}+\frac{4905}{512}M_1^2\right)\cos 2\tau \\
\qquad -\frac{16973}{256}M_1\cos 3\tau+\left(\frac{23051}{8.3.25}+\frac{21}{32}M_1^2\right)\cos 4\tau \\
\qquad +\frac{1665}{512}M_1\cos 5\tau+\frac{1393}{256}\cos 6\tau, \\
Z_7^{(c)} = -\left(\frac{1491347}{1024}+\frac{315}{128}M_1^2\right)M_1\cos\tau \\
\qquad +\left(\frac{759413}{512.81.5}+\frac{59445}{1024}M_1^2\right)\cos 2\tau-\frac{1235971}{4096}M_1\cos 3\tau \\
\qquad +\left(\frac{97051}{16.125}+\frac{10763}{1024.5}M_1^2\right)\cos 4\tau-\frac{10077}{1024}M_1\cos 5\tau \\
\qquad +\frac{562041}{512.35}\cos 6\tau \\
\qquad \ldots\ldots\ldots\ldots
\end{cases}
$$

If we consider m_0 as fixed, where it appears in c^2, and as a variable parameter elsewhere, then, since c^2 is not an integer for $|m_0|$ sufficiently small and distinct from zero, (37) will belong to Case I. In the end the m_0 where it occurs as a parameter must have the same value as it has where it occurs in c^2 in order that the result shall be a solution of the physical problem.

According to § 6, p. 352, *loc. cit.*, the general solution of (37) is then

of the form

$$(40)\quad \begin{cases} Z = c_1 e^{\sigma\sqrt{-1}\,\tau}\,\xi^{(1)}(m_0\,;\,\tau) + c_2 e^{-\sigma\sqrt{-1}\,\tau}\,\xi^{(2)}(m_0\,;\,\tau), \\ \text{where } c_1,\ c_2 \text{ are arbitrary constants, and} \\ \xi^{(i)} = \sum_{=0}^{\infty} \xi_j^{(i)}(\tau)\, m_0^j, \quad \xi_j^{(i)}(\tau+2\pi) = \xi_j^{(i)}(\tau) \quad (i=1,2\,;\ j=0,\ldots,\infty), \\ \sigma = c + \sum_{j=1}^{\infty} \sigma_j m_0^j, \\ \xi_0^{(i)} = 1, \quad \xi_j^{(i)} = 0 \quad (i=1,2\,;\ j=1,\ \ldots,\ \infty). \end{cases}$$

The σ_j are all real, and the $\xi^{(i)}$ when expanded as Fourier series have the form

$$(41)\quad \begin{cases} \xi^{(1)} = \sum_{j=0}^{\infty} (C_j^{(1)} \cos j\tau + \sqrt{-1}\, C_j^{(2)} \sin j\tau), \\ \xi^{(2)} = \sum_{j=0}^{\infty} (C_j^{(1)} \cos j\tau - \sqrt{-1}\, C_j^{(2)} \sin j\tau). \end{cases}$$

It is only necessary to construct one particular solution, for the other one can be obtained from it by merely changing the sign of $\sqrt{-1}$. The differential equation is unchanged by changing the signs of both c and τ, and it therefore follows that the σ_j and $C_j^{(2)}$ are odd in c.

The formulæ for the construction of the successive coefficients in (40) are given in § 7, *loc. cit.* In these formulæ there arise quadratures of the form

$$\int \begin{matrix}\sin\\ \cos\end{matrix} j\tau\, e^{2c\sqrt{-1}\,\tau}\, d\tau = \frac{e^{2c\sqrt{-1}\,\tau}}{j^2-4c^2} \left\{ 2c\sqrt{-1} \begin{matrix}\sin\\ \cos\end{matrix} j\tau \mp j \begin{matrix}\cos\\ \sin\end{matrix} j\tau \right\},$$

and, in order to simplify the computation, the abridged notation

$$D_j = \frac{1}{j^2-4c^2}$$

has here been introduced for the divisors. This also facilitates checking the results, for the solutions then satisfy the differential equations identically in the D_j.

It might be supposed that the solution by this method breaks down when $|m_0|$ is very small, since D_2 then becomes very large, but this is not the case. When the generalisation of m_0 was made, we said that the differential equation belonged to Case I when $|m_0|$ is sufficiently small but distinct from zero; but as a matter of fact, on account of the special character of (39), we are able to solve (38) according to the method of Case I no matter

how small $|m_0|$ may be, the zero value included. For, by (39), since c^2 contains a linear term,

$$(42) \qquad D_2 = \frac{1}{4(1-c^2)} = \tfrac{1}{4}\{2m_0-(1+\tfrac{3}{2}\eta)\,m_0^2+\ldots\}^{-1} = \frac{1}{m_0}d_2(m_0),$$

where $d_2(0) \neq 0$, and, since the explicit coefficient of m_0 in (38) is zero, D_2 does not occur in the solutions except when multiplied by a second or higher power of m_0. Therefore the objection to the generalisation of m_0 and to the use of the method of Case I for solution, when $|m_0|$ is very small, is only an apparent one, on account of the property of this special case.

It further follows from (42) that the divisor D_2 decreases the order to which the solutions are carried by one. This peculiarity is not due to the method of treatment given in this paper, but seems to be inherent in the problem, as in every treatment so far given of the lunar theory the divisors have reduced the order of the solutions by one.

The first particular solution of (38) has been constructed to the seventh power of m_0 inclusive, and the results are here given :*

$$(43) \qquad \begin{cases} \sigma_1 = \sigma_2 = \sigma_3 = 0, \quad \sigma_4 = \dfrac{9}{4c}D_2, \quad \sigma_5 = \dfrac{57}{4c}D_2, \\ \sigma_6 = \dfrac{681}{16c}D_2 + \dfrac{2025}{1024c}M_1^2 D_1, \\ \sigma_7 = \dfrac{1}{c}\left\{\dfrac{675}{32}M_1^2 D_1 + \dfrac{233}{3}D_2 + \dfrac{405}{32}M_1 D_1 D_2\right\}, \end{cases}$$

(44)

$$\begin{cases} \xi_0^{(1)} = 1, \quad \xi_1^{(1)} = 0, \\ \xi_2^{(1)} = 3D_2\{(\cos 2\tau - 1) - \sqrt{-1}\, c \sin 2\tau\}\,\eta, \\ \xi_3^{(1)} = \left[-\dfrac{45}{16}M_1 D_1(\cos\tau - 1) + \dfrac{19}{2}D_2(\cos 2\tau - 1)\right. \\ \qquad\qquad \left. + \sqrt{-1}\, c\left\{\dfrac{45}{8}M_1 D_1 \sin\tau - \dfrac{19}{2}D_2 \sin 2\tau\right\}\right]\eta, \end{cases}$$

* Since η is very nearly equal to unity in the application to the lunar theory, it has been retained only to the third coefficient, no sensible error being thus introduced.

(44)—*continued.*

$$\xi_4^{(1)} = -15M_1D_1(\cos\tau-1) + \frac{D_2}{3}(40-27D_2)(\cos 2\tau-1)$$
$$+\frac{75}{64}M_1D_3(\cos 3\tau-1)$$
$$+\left[\frac{3}{8}(11+36D_2)D_4-\frac{9}{16}D_2\right](\cos 4\tau-1)$$
$$+\sqrt{-1}\,c\left\{30M_1D_1\sin\tau+\frac{D_2}{3}(27D_2-40)\sin 2\tau-\frac{25}{32}M_1D_3\sin 3\tau\right.$$
$$\left.-\frac{3}{16}(11+36D_2)D_4\sin 4\tau\right\}$$

$$\xi_5^{(1)} = \left[-\frac{19491}{256}D_1+\frac{135}{64}D_2(D_1+1)\right]M_1(\cos\tau-1)$$
$$+\left[\frac{86}{9}-57D_2+\frac{135}{16}M_1D_1\right]D_2(\cos 2\tau-1)$$
$$+\left[\frac{45}{64}(2D_1+D_2)-D_3\left\{\frac{705}{128}+\frac{135}{64}(8D_1+5D_2)\right\}\right]M_1(\cos 3\tau-1)$$
$$+\left[\frac{2937}{160}D_4-\frac{57}{16}D_2(1-24D_4)\right](\cos 4\tau-1)$$
$$+\sqrt{-1}\,c\left\{\left[\frac{18411}{128}-\frac{135}{32}D_2\right]M_1D_1\sin\tau\right.$$
$$+\left[-\frac{86}{9}+57D_2-\frac{135}{16}M_1D_1\right]D_2\sin 2\tau$$
$$+\left[\frac{235}{64}+\frac{45}{32}(8D_1+5D_2)\right]M_1D_3\sin 3\tau$$
$$\left.-\left[\frac{2937}{320}+\frac{171}{4}D_2\right]D_4\sin 4\tau\right\},$$

$$\xi_6^{(1)} = \left[-\frac{347045}{1024}D_1+\frac{4815}{256}D_2(D_1+1)-\frac{2025}{256}M_1^2D_1^2\right.$$
$$\left.+\frac{75}{32}D_1D_3-\frac{75}{128}D_3\right]M_1(\cos\tau-1)$$

(44)—*continued.*

$$+\left[\left(\frac{29401}{256.27}+\frac{4905}{512}M_1^2\right)D_2+27D_2^3-\frac{5313}{32}D_2^2\right.$$
$$+\frac{6075}{512}D_1D_2M_1^2-\frac{9}{128}(11+36D_2)(4D_2+1)D_4-\frac{2025}{1024}M_1^2D_1$$
$$\left.+\left(\frac{2295}{32}D_1-\frac{225}{64}D_3\right)M_1D_2\right](\cos 2\tau-1)$$
$$+\left[-\left(\frac{16973}{256}+\frac{2295}{16}D_1+\frac{11925}{128}D_2\right)D_3\right.$$
$$\left.+\frac{765}{128}(2D_1+D_2)\right]M_1(\cos 3\tau-1)$$
$$+\left[\left(\frac{23051}{8.3.25}+\frac{21}{32}M_1^2\right)D_4+243D_2D_4-\frac{81}{2}D_2^2D_4\right.$$
$$\left.-\frac{1}{64}(681-108D_2)D_2\right](\cos 4\tau-1)$$
$$+\left[\left(\frac{1665}{512}+\frac{1575}{256}D_2+\frac{75}{16}D_3\right)D_5-\frac{45}{246}D_2-\frac{15}{128}D_3\right]M_1(\cos 5\tau-1)$$
$$+\left[\left(\frac{1393}{256}+\frac{765}{32}D_2\right)D_6+\frac{45}{32}(11+36D_2)D_4D_6-\frac{33}{64}D_2\right.$$
$$\left.-\frac{3}{128}(11+36D_2)D_4\right](\cos 6\tau-1)$$
$$+\sqrt{-1}\,c\left\{\left[\frac{310325}{512}-\frac{4815}{128}D_2-\frac{75}{16}D_3+\frac{2025}{128}D_1M_1^2\right]M_1D_1\sin\tau\right.$$
$$+\left[-\left(\frac{18709}{256.27}+\frac{4905}{512}M_1^2\right)-\frac{6075}{512}M_1^2D_1\right.$$
$$-\left(\frac{2295}{32}D_1-\frac{225}{64}D_3\right)M_1+\frac{5421}{32}D$$
$$\left.+\frac{9}{32}(11+36D_2)D_4\right]D_2\sin 2\tau$$
$$+\left[\frac{16973}{384}+\frac{765}{8}D_1+\frac{3975}{64}D_2\right]M_1D_3\sin 3\tau$$
$$+\left[-\left(\frac{23051}{16.3.25}+\frac{21}{64}M_1^2\right)-\frac{243}{2}D_2+\frac{81}{4}D_2^2\right]D_4\sin 4\tau$$
$$-\left[\frac{333}{256}+\frac{315}{128}D_2+\frac{15}{8}D_3\right]M_1D_5\sin 5\tau$$
$$\left.-\left[\frac{1393}{768}+\frac{255}{32}D_2+\frac{15}{32}(11+36D_2)D_4\right]D_6\sin 6\tau\right\},$$

(44)—*continued.*

$$
\begin{aligned}
\overset{(1)}{\gamma} = \Big[& -\Big(\frac{1693973}{1024}+\frac{315}{128}M_1^2\Big)D_1-\frac{531}{128}D_1^2+\frac{112353}{1024}D_1D_2 \\
& -\frac{405}{64}D_1D_2(4D_1-D_2)-\frac{675}{8}M_1D_1^2+\frac{115}{128}D_3+\frac{99933}{1024}D_2 \\
& +\frac{3375}{1024}M_1D_1D_3+\frac{135}{128}(11+36D_2)D_1D_4 \\
& -\frac{135}{128}(8D_1+5D_2)(4D_1-1)D_3-\frac{115}{32}D_1D_3 \\
& -\frac{405}{64}D_2^2\Big]M_1(\cos\tau-1) \\
+\Big[& \Big(\frac{640507}{512.81}+\frac{59445}{1024}M_1^2\Big)D_2-\frac{51953}{64.3}D_2^2+\frac{89775}{4096}M_1^2D_1 \\
& -\frac{5625}{2048}M_1^2D_2D_3+\frac{345}{64}M_1D_2D_3-\frac{2025}{64}M_1D_1D_2^2-\frac{135}{16}M_1D_2^2 \\
& +\frac{405}{64}(8D_1+5D_2)M_1D_2D_3+\frac{1125}{4096}M_1^2D_3-\frac{1539}{64}D_2D_4(4D_2+1) \\
& +\frac{103473}{256}M_1D_1D_2+\frac{262575}{2048}M_1^2D_1D_2\Big](\cos 2\tau-1) \\
+\Big[& \frac{33771}{1024}D_2-\frac{1235971}{4096}D_3+\frac{32781}{512}D_1-\frac{496995}{1024}D_2D_3 \\
& -\frac{50319}{64}D_1D_3-\frac{135}{128}D_1D_2-\frac{135}{64}D_2^2+\frac{405}{32}D_1D_2D_3+\frac{2025}{64}D_2^2D_3 \\
& -\frac{945}{1024}(11+36D_2)D_3D_4+\frac{45}{1024}(11+36D_2)D_4 \\
& +\frac{3375}{1024}M_1D_1D_3\Big]M_1(\cos 3\tau-1) \\
+\Big[& \Big(\frac{97051}{16.125}+\frac{10763}{1024.5}M_1^2\Big)D_4-\frac{233}{12}D_2+\frac{513}{32}D_2^2+\frac{178437}{32.3.5}D_2D_4 \\
& -\frac{1539}{4}D_2^2D_4-\frac{16875}{2048}M_1^2D_1D_4-\frac{7875}{2048}M_1^2D_3D_4+\frac{1125}{8192}M_1^2D_3 \\
& +\frac{3375}{8192}M_1^2D_1-\frac{405}{256}M_1D_1D_2+\frac{1485}{128}M_1D_1D_4 \\
& +\frac{1215}{32}M_1D_1D_2D_4\Big](\cos 4\tau-1)
\end{aligned}
$$

(44)—*continued.*

$$+\Big[-\frac{10077}{1024}D_5+\frac{297}{256}D_1+\frac{69}{256}D_2+\frac{23}{128}D_3-\frac{3885}{512}D_2D_5$$
$$-\frac{4185}{128}D_1D_5-\frac{115}{16}D_3D_5-\frac{1215}{1024}(11+36D_2)D_4D_5$$
$$-\frac{135}{16}(8D_1+5D_2)D_3D_5+\frac{27}{1024}(11+36D_2)D_4$$
$$+\frac{27}{128}(8D_1+5D_2)D_3\Big]M_1(\cos 5\tau-1)$$

$$+\Big[\frac{562041}{512.35}D_6+\frac{57759}{320}D_2D_6+\frac{15081}{80}D_4D_6-\frac{5027}{1280}(D_2+D_4)$$
$$-\frac{513}{64}D_2D_4+\frac{7695}{16}D_2D_4D_6\Big](\cos 6\tau-1)$$

$$+\sqrt{-1}\,c\Big\{\Big[\Big(\frac{1491347}{512}+\frac{315}{64}M_1^2\Big)-\frac{102633}{512}D_2+\frac{531}{64}D_1$$
$$+\frac{675}{4}M_1D_1+\frac{405}{32}D_2^2-\frac{3375}{512}M_1D_3$$
$$-\frac{135}{64}(11+36D_2)D_4+\frac{115}{16}D_3+\frac{135}{16}(8D_1+5D_2)D_3$$
$$-\frac{405}{8}D_1D_2\Big]M_1D_1\sin\tau$$

$$+\Big[-\Big(\frac{759413}{512.81.5}+\frac{59445}{1024}M_1^2\Big)+\frac{58109}{62.3}D_2$$
$$-\frac{103473}{256}M_1D_1-\frac{262575}{2048}M_1^2D_1+\frac{5625}{2048}M_1^2D_2$$
$$+\frac{1539}{16}D_2D_4-\frac{345}{64}M_1D_3+\frac{2025}{64}M_1D_1D_2+\frac{135}{16}M_1D_2$$
$$-\frac{405}{64}(8D_1+5D_2)M_1D_3\Big]D_2\sin 2\tau$$

$$+\Big[\frac{1235971}{2048.3}+\frac{165665}{512}D_2+\frac{16773}{32}D_1-\frac{1125}{512}M_1D_1$$
$$-\frac{675}{32}D_2^2-\frac{135}{16}D_1D_2+\frac{315}{512}(11+36D_2)D_4\Big]M_1D_3\sin 3\tau$$

(44)—*continued.*

$$+\Big[-\Big(\frac{97051}{32.125}+\frac{10763}{2048.5}M_1^2\Big)-\frac{178437}{64.3.5}D_2+\frac{1539}{8}D_2^2$$
$$-\frac{1485}{256}M_1D_1+\frac{16875}{4096}M_1^2D_1+\frac{7875}{4096}M_1^2D_3$$
$$-\frac{1215}{64}M_1D_1D_2\Big]D_4\sin 4\tau$$
$$+\Big[\frac{10077}{512.5}+\frac{777}{256}D_2+\frac{837}{64}D_1+\frac{23}{8}D_3$$
$$+\frac{243}{512}(11+36D_2)+\frac{27}{8}(8D_1+5D_2)D_3\Big]M_1D_5\sin 5\tau$$
$$+\Big[-\frac{187347}{512.35}-\frac{19253}{320}D_2-\frac{5027}{80}D_4$$
$$-\frac{2565}{16}D_2D_4\Big]D_6\sin 6\tau\Big\}$$

7. *Conditions for a Symmetric Periodic Solution of* (19).

By property (γ) of the differential equations, the right hand side of the third equation of (19) is odd in z, and it is therefore plain that in the z there will be no term independent of $e^{\sigma\sqrt{-1}\,\tau}$ or $e^{-\sigma\sqrt{-1}\,\tau}$. These exponentials are rotating factors which correspond to the regression of the line of nodes, and in order that the solution obtained shall satisfy the physical problem it is necessary to determine the m_0 so that the rate of regression given by the exponentials shall be equal to the rate of regression given by observation.

The solution of the first degree terms in z is

$$z_1 = c_1^{(1)}e^{\sigma\sqrt{-1}\,\tau}\,\zeta^{(1)}(m_0\,;\,\tau)+c_2^{(1)}e^{-\sigma\sqrt{-1}\,\tau}\,\zeta^{(2)}(m_0\,;\,\tau),$$

and these are relatively the most important terms in the solution when $c_1^{(1)}$ and $c_2^{(1)}$ are small. The period of the exponentials in z is $2\pi/\sigma$, and the $\zeta^{(i)}$ are periodic with the period 2π. Then in order that z shall be periodic these two periods must be commensurable; that is, σ must be a rational number. Since $z(0) = 0$, it follows from (40) that $c_1^{(1)}+c_2^{(1)} = 0$, and thence that $z(\pi/\sigma)$ and $z(2\pi/\sigma)$ are relatively very small. This means that at $\tau = 2\pi/\sigma$ the infinitesimal body is in the neighbourhood of the

node from which it started, and, since the period of the solution is to be a multiple of 2π, it must return to the node after an integral number of synodic revolutions. Therefore the mean period of an oscillation from the plane of motion of the finite bodies is

$$\frac{2\pi}{\sigma}.$$

Now we supposed at the beginning of this paper that the period of regression of the line of nodes and the synodic period of the three bodies are given by observation. Let them be denoted by T_1 and T_2 respectively. Then T_1 and T_2 are rational numbers, and we can determine from them the mean rate of synodic revolution and the mean rate of regression of the line of nodes; the former, in our notation, is $n-n'$ in t or unity in τ, and we shall denote the latter by q. Thus, with respect to axes rotating with uniform angular velocity equal to n in t or m in τ, the mean rate of regression of the line of nodes is $m+q$. Therefore, when $\tau = \frac{2\pi}{1+m+q}$, the infinitesimal body is in the neighbourhood of the node from which it started, and, since in numerical applications the sexagesimal system of measuring angles is used, m and q are rational numbers. Therefore the mean period of oscillation from the plane of motion of the finite bodies is

$$\frac{2\pi}{1+m+q} = \frac{2\nu\pi}{N},$$

where ν and N are relatively prime integers.

In order, then, that the solution shall represent the physical problem, we must have

$$\sigma = 1+m+q = \frac{N}{\nu}. \tag{45}$$

On substituting for c in (43) its expression as a power series in m_0, given in (39), we obtain for the expansion of σ,

$$\sigma = 1+m_0+\frac{3}{4}\eta m_0^2-\left(\frac{9}{32}+\frac{3}{4}\eta\right)m_0^3-\frac{105}{128}m_0^4 + \frac{31}{512}m_0^5+\left(\frac{611}{1536}+\frac{135}{256}M_1+\frac{3555}{1024}M_1^2\right)m_0^6+\ldots. \tag{46}$$

When σ has been replaced by its expanded form, equation (45) becomes

$$-(m+q)+m_0+\frac{3}{4}\eta m_0^2-\left(\frac{9}{32}+\frac{3}{4}\eta\right)m_0^3+\ldots = 0,$$

and, by letting $m+q = l$, this equation again becomes

$$-l+m_0+\frac{3}{4}\eta m_0^2-\left(\frac{9}{32}+\frac{3}{4}\eta\right)m_0^3+\ldots=0. \tag{47}$$

Equation (47) is analytic in m_0 and l; the left hand side vanishes for $m_0 = l = 0$; and the coefficient of the linear term in m_0 is distinct from zero. Therefore, by the theory of implicit functions, m_0 is uniquely determined by (47) as a power series in l, convergent when $|l|$ is sufficiently small. The solution of (47) is

$$m_0 = l-\frac{3}{4}\eta l^2+\left(\frac{9}{32}+\frac{3}{4}\eta+\frac{9}{8}\eta^2\right)l^3-\frac{165}{32}l^4 \tag{48}$$
$$+\frac{16435}{1024}l^5-\left(\frac{615421}{4096.3}+\frac{135}{256}M_1+\frac{3555}{1024}M_1^2\right)l^6+\ldots.$$

When m_0 has been determined by (48), the exponentials $e^{\pm\sigma\sqrt{-1}\,\tau}$ give the observed rate of regression of the nodes; and it will now be shown that it is possible to construct a solution of (19) which is symmetrical and periodic with the period $2\nu\pi$.

8. *The Preliminary Steps of the Construction of the Solution of* (19).

In § 3 reference was made to an existence theorem for a class of differential equations by Prof. MacMillan by means of which the existence and convergence of periodic solutions as power series in a parameter are assured if it is possible to construct them formally. Equations (14) were shown to fulfil the hypotheses of the theorem, and since (19) differ from (14) only by the transformation (18), it is readily seen that the differential equations (19) also satisfy the hypotheses. By general theory it is necessary to investigate certain determinants in order to ascertain in what fractional power of the parameter the solutions are expansible, but in the present case one can see intuitively what form the solution ought to have.

From the form of the initial conditions (10) in Prof. Moulton's memoir, *loc. cit.*, § 3, it is seen that at the initial time the infinitesimal body is in conjunction at its apse. And by the initial conditions (15), it is also passing through one of the nodes. But since it might be passing through either node at the initial time, this ought to manifest itself in the solution by the z being determined except for sign, and the r and θ being uniquely determined. This suggests that p and u are expansible as power series in integral powers of λ, and z as a power series in odd powers of $\lambda^{\frac{1}{2}}$.

We therefore assume the solution

$$(49) \qquad p = \sum_{j=1}^{\infty} p_{2j}\lambda^j, \quad u = \sum_{j=1}^{\infty} u_{2j}\lambda^j, \quad z = \sum_{j=1}^{\infty} z_{2j-1}\lambda^{\frac{1}{2}(2j-1)}.$$

Since this solution is to be periodic with the period $2\nu\pi$ whenever $|\lambda|$ is sufficiently small, it follows that the p_{2j}, u_{2j}, z_{2j-1} must be separately periodic with the same period. Also, since $\dot{p}(0) = u(0) = z(0) = 0$, it follows that

$$\dot{p}_{2j}(0) = u_{2j}(0) = z_{2j-1}(0) = 0 \quad (j = 1, \ldots, \infty),$$

The First Step.—The coefficient of $\lambda^{\frac{1}{2}}$ is defined by the equation

$$(50) \qquad z_1 + \psi^{(c)}(m_0\,;\,\tau)\,z_1 = 0,$$

the general solution of which is

$$(51) \qquad z_1 = c_1^{(1)} e^{\sigma\sqrt{-1}\,\tau}\,\xi^{(1)}(m_0\,;\,\tau) + c_2^{(1)} e^{-\sigma\sqrt{-1}\,\tau}\,\xi^{(2)}(m_0\,;\,\tau),$$

where $c_1^{(1)}$ and $c_2^{(1)}$ are arbitrary constants, and the forms of σ and $\xi^{(i)}$ are given in (40) and (41). Since $z_1(0) = 0$, $c_1^{(1)}$ and $c_2^{(1)}$ must satisfy the relation

$$(52) \qquad c_1^{(1)} + c_2^{(1)} = 0,$$

which leaves one arbitrary constant.

The Second Step.—The coefficients of λ are defined by the equations

$$(53) \qquad \begin{cases} \ddot{p}_2 + \theta_1^{(c)}(m_0\,;\,\tau)\,p_2 + \theta_4^{(s)}(m_0\,;\,\tau)\,u_2 + \theta_5^{(c)}(m_0\,;\,\tau)\,\dot{u}_2 = P_2, \\ r_0\,\ddot{u}_2 + \phi_1^{(s)}(m_0\,;\,\tau)\,p_2 + \phi_2^{(c)}(m_0\,;\,\tau)\,\dot{p}_2 + \phi_5^{(c)}(m_0\,;\,\tau)\,u_2 = Q_2, \end{cases}$$

where, by the properties (α), (β), (γ), (δ) of (19),

$$(53a) \qquad P_2 = P_2^{(c)}(m_0\,;\,\tau)\,z_1^2, \quad Q_2 = Q_2^{(s)}(m_0\,;\,\tau)\,z_1^2.$$

From the forms of $\xi^{(i)}$ in (41), and the forms of P_2 and Q_2 in (53a), it follows that P_2 and Q_2 have the following properties:

(a') P_2 and Q_2 are homogeneous polynomials of the second degree in $c_1^{(1)} e^{\sigma\sqrt{-1}\,\tau}$ and $c_2^{(1)} e^{-\sigma\sqrt{-1}\,\tau}$ whose coefficients are sine and cosine series of integral multiples of τ.

(b') P_2 and Q_2 are unchanged by interchanging $c_1^{(1)}$ and $c_2^{(1)}$ and changing the sign of $\sqrt{-1}$.

(c') In the coefficients of P_2, the coefficients of the cosine terms are real, and the coefficients of the sine terms are purely imaginary.

(d') In the coefficients of Q_2, the coefficients of the sine terms are real, and the coefficients of the cosine terms are purely imaginary.

(e') The coefficients of $c_1^{(1)}$, $c_2^{(1)}$ n P_2 and Q_2 are cosine and sine series respectively.

The general solution of the left members of (53) set equal to zero, by (36), is

(54)

$$\begin{cases} p_2 = a_1 e^{\omega\sqrt{-1}\,\tau}\,\xi^{(1)}(\tau) + a_2 e^{-\omega\sqrt{-1}\,\tau}\,\xi^{(2)}(\tau) + a_3\psi_1^{(s)}(\tau) + a_4\{\chi_1^{(c)}(\tau) + \tau\psi_1^{(s)}(\tau)\}, \\ u_2 = a_1 e^{\omega\sqrt{-1}\,\tau}\,\eta^{(1)}(\tau) + a_2 e^{-\omega\sqrt{-1}\,\tau}\,\eta^{(2)}(\tau) + a_3\psi_3^{(c)}(\tau) + a_4\{\chi_3^{(s)}(\tau) + \tau\psi_3^{(c)}(\tau)\}, \end{cases}$$

where $a_1, \ldots, a_4$ are arbitrary constants. The general solution of (53) is found from (54) by the method of variation of parameters.

Since the exponentials in (54) are not in general periodic with the period $2\nu\pi$, the complementary functions arising from the first two parts of (54) are excluded by assigning zero values to the arbitrary constants which they contain as factors. Since interchanging the $c_1^{(1)}$ and $c_2^{(1)}$, and changing the sign of $\sqrt{-1}$, leaves the differential equations unchanged, and changing the signs of u_2, τ, and $\sqrt{-1}$ also leaves them unchanged, it follows that the particular integrals arising from the first two parts of (54) have the properties (a'), ..., (e'), when P_2 and Q_2 are replaced by p_2 and u_2 respectively.

When (54) are substituted in (53), the differential equations which determine a_3 and a_4, so that (54) shall be a solution of (53), are

(55)

$$\Delta_1\dot{a}_3 = \begin{vmatrix} \xi^{(1)}, & \xi^{(2)}, & 0, & \chi_1^{(c)} + \tau\psi_1^{(s)} \\ \omega\sqrt{-1}\,\xi^{(1)} + \dot{\xi}^{(1)}, & -\omega\sqrt{-1}\,\xi^{(2)} + \dot{\xi}^{(2)}, & P_2, & \dot{\chi}_1^{(c)} + \psi_1^{(s)} + \tau\dot{\psi}_1^{(s)} \\ \eta^{(1)}, & \eta^{(2)}, & 0, & \chi_3^{(s)} + \tau\psi_3^{(c)} \\ \omega\sqrt{-1}\,\eta^{(1)} + \dot{\eta}^{(1)}, & -\omega\sqrt{-1}\,\eta^{(2)} + \dot{\eta}^{(2)}, & Q_2, & \dot{\chi}_3^{(s)} + \psi_3^{(c)} + \tau\dot{\psi}_3^{(c)} \end{vmatrix},$$

$$\Delta_1\dot{a}_4 = \begin{vmatrix} \xi^{(1)}, & \xi^{(2)}, & \psi_1^{(s)}, & 0 \\ \omega\sqrt{-1}\,\xi^{(1)} + \dot{\xi}^{(1)}, & -\omega\sqrt{-1}\,\xi^{(2)} + \dot{\xi}^{(2)}, & \dot{\psi}_1^{(s)}, & P_2 \\ \eta^{(1)}, & \eta^{(2)}, & \psi_3^{(c)}, & 0 \\ \omega\sqrt{-1}\,\eta^{(1)} + \dot{\eta}^{(1)}, & -\omega\sqrt{-1}\,\eta^{(2)} + \dot{\eta}^{(2)}, & \dot{\psi}_3^{(c)}, & Q_2 \end{vmatrix}.$$

where Δ_1 is the determinant of the fundamental set of solutions (22) and (35), and is a constant different from zero. From (24) and (53a) it follows that the first explicit determinant in (55) is an even function of τ, and therefore (55) may be written in the form

$$(56) \qquad \begin{cases} \dot{a}_3 = A^{(2)} + \psi_1^{(2)}(\tau) + \tau\psi_2^{(2)}(\tau), \\ \dot{a}_4 = \qquad\qquad - \psi_2^{(2)}(\tau), \end{cases}$$

where $A^{(2)}$ is a constant, and $\psi_1^{(2)}$, $\psi_2^{(2)}$ are even and odd functions of τ respectively. From the forms (25) of $\xi^{(i)}$ and $\eta^{(i)}$ $(i = 1, 2)$, it follows that Δ_1 is purely imaginary. The first determinant of (55) changes sign by changing the signs of both τ and $\sqrt{-1}$, and therefore $\psi_1^{(2)}$ and $\psi_2^{(2)}$ have the fundamental properties (a'), ..., (e') of P_2 and Q_2 respectively.

The integration of (56) gives results of the form

$$(57) \qquad \begin{cases} a_3 = A^{(2)}\tau + \chi_1^{(2)}(\tau) - \tau\chi_2^{(2)}(\tau) + a_3^{(2)}, \\ a_4 = \qquad\qquad\qquad\qquad \chi_2^{(2)}(\tau) + a_4^{(2)}, \end{cases}$$

where $a_3^{(2)}$ and $a_4^{(2)}$ are arbitrary constants, and $\chi_1^{(2)}$, $\chi_2^{(2)}$ have the fundamental properties of Q_2 and P_2. The parts of the general solution of (53) arising from the last two parts of (54) then become

$$(58) \qquad \begin{cases} p_2 = (A^{(2)}\tau + \chi_1^{(2)}(\tau) - \tau\chi_2^{(2)}(\tau) + a_3^{(2)})\,\psi_1^{(s)} + (\chi_2^{(2)} + a_4^{(2)})(\chi_1^{(c)} + \tau\psi_1^{(s)}), \\ u_2 = (A^{(2)}\tau + \chi_1^{(2)}(\tau) - \tau\chi_2^{(2)}(\tau) + a_3^{(2)})\,\psi_3^{(c)} + (\chi_2^{(2)} + a_4^{(2)})(\chi_3^{(s)} + \tau\psi_3^{(c)}). \end{cases}$$

In order that (58) shall be symmetric and periodic, we must put

$$(59) \qquad a_3^{(2)} = 0, \qquad a_4^{(2)} = -A^{(2)}.$$

Thus all the constants of integration arising at this step have been determined, and the complete symmetric periodic solution of (53) has the fundamental properties (a'), ..., (e') of P_2 and Q_2 respectively.

The Third Step.—The coefficient of $\lambda^{\frac{3}{2}}$ is defined by

$$(60) \qquad \begin{cases} \ddot{z}_3 + \psi_3^{(c)}(m_0\,;\,\tau)z_3 = R_3, \\ R_3 = R_{3,1}^{(c)}(\tau)z_1 + R_{3,2}^{(c)}(\tau)z_1^3 + R_{3,3}^{(c)}(\tau)p_2z_1 + R_{3,1}^{(s)}(\tau)u_2z_1. \end{cases}$$

From the properties of z_1, p_2, u_2, and the nature of the coefficients in R_3, it follows that R_3 has the following properties:—

(a'') It is an odd degree polynomial in $c_1^{(1)}e^{\sigma\sqrt{-1}\,\tau}$ and $c_2^{(1)}e^{-\sigma\sqrt{-1}\,\tau}$, whose coefficients are sine and cosine series of integral multiples of τ.

(b'') It is unchanged by interchanging $c_1^{(1)}$ and $c_2^{(1)}$ and changing the sign of $\sqrt{-1}$.

(c'') In its coefficients, the coefficients of the cosine terms are real, and the coefficients of the sine terms are purely imaginary.

The general solution of the left member of (60) set equal to zero is

$$z_3 = C_1^{(3)} e^{\sigma\sqrt{-1}\,\tau}\,\xi^{(1)}(m_0;\tau) + C_2^{(3)} e^{-\sigma\sqrt{-1}\,\tau}\,\xi^{(2)}(m_0;\tau), \tag{61}$$

where $C_1^{(3)}$, $C_2^{(3)}$ are arbitrary constants. In order that (61) shall be a solution of the complete equations (60), $C_1^{(3)}$ and $C_2^{(3)}$ must satisfy the differential equation

$$\begin{cases} \Delta_2 \dot{C}_1^{(3)} = -\xi^{(2)} R_3 e^{-\sigma\sqrt{-1}\,\tau}, \\ \Delta_2 \dot{C}_2^{(3)} = \quad \xi^{(1)} R_3 e^{\sigma\sqrt{-1}\,\tau}, \end{cases} \tag{62}$$

where Δ_2 is the determinant of the fundamental set of solutions of the left member of (60) set equal to zero, and is a constant different from zero. But if $C_1^{(3)}$ and $C_2^{(3)}$ are to be periodic, the right members of (62) must contain no constant terms.

From the forms of the $\xi^{(i)}$ given in (41), it follows that the constant parts of (62), arising from the first term of R_3, are the constant parts of the products

$$-c_1^{(1)}\xi^{(1)}\xi^{(2)} R_{3,1}^{(c)}(\tau), \qquad c_2^{(1)}\xi^{(1)}\xi^{(2)} R_{3,1}^{(c)}(\tau).$$

Since

$$\xi^{(1)}(\sqrt{-1}) = \xi^{(2)}(-\sqrt{-1}),$$

the constant parts of these products are of the form

$$-c_1^{(1)} P_1^{(3)}(m_0), \qquad c_2^{(1)} P_1^{(3)}(m_0),$$

where $P_1^{(3)}$ is a power series in m_0 with real coefficients.

The constant parts of (62) arising from the second part of R_3 are the constant parts of the products

$$-(c_1^{(1)})^2 c_2^{(1)} (\xi^{(1)}\xi^{(2)})^2 R_{3,2}^{(c)}(\tau), \qquad (c_2^{(1)})^2 c_1^{(1)} (\xi^{(1)}\xi^{2)})^2 R_{3,2}^{(c)}(\tau);$$

and are therefore of the form

$$-(c_1^{(1)})^2 c_2^{(1)} P_{3,2}^{(3)}(m_0), \qquad (c_2^{(1)})^2 c_1^{(1)} P_{3,2}^{(3)}(m_0), \tag{63}$$

where $P_{3,2}^{(3)}$ is a power series in m_0.

The constant parts of (62) arising from the last two parts of R_3 are easily seen to be of the same form as (63). Therefore, in order that the solutions of (62) shall be periodic, $c_1^{(1)}$ and $c_2^{(1)}$ must satisfy conditions of the form

$$c_1^{(1)} P_1^{(3)}(m_0) + (c_1^{(1)})^2 c_2^{(1)} P_2^{(3)}(m_0) = 0,$$

$$c_2^{(1)} P_1^{(3)}(m_0) + (c_2^{(1)})^2 c_1^{(1)} P_2^{(3)}(m_0) = 0,$$

and, since $c_1^{(1)} = -c_2^{(1)}$, these two conditions are identical. They are satisfied by $c_1^{(1)} = c_2^{(1)} = 0$, but in that case, by the properties (γ) and (δ) of the differential equations and by the process of solving (53), it follows that the solution reduces to the trivial one

$$p \equiv u \equiv z \equiv 0.$$

They are also satisfied by

$$c_1^{(1)} = -c_2^{(1)} = \pm \sqrt{P_1^{(3)}/P_2^{(3)}}. \tag{64}$$

In order that (64) shall determine $c_1^{(1)}$, it is necessary to show that $P_1^{(3)}$ and $P_2^{(3)}$ are not zero. An examination of the original differential equations (6) shows that $R_{3,1}^{(c)}$ is distinct from zero for $m_0 = 0$, and since

$$\xi^{(1)}(0\,;\ \tau) = \xi^{(2)}(0\,;\ \tau) = 1,$$

it follows that $P_1^{(3)}(0)$ is distinct from zero, and therefore $P_1^{(3)}$ is distinct from zero when $|m_0|$ is sufficiently small. A similar investigation, however, shows that $P_2^{(3)}(0) = 0$. By equation (47), m_0 can only vanish with l, that is, with m and q. For the physical problem, q vanishes with m, and the vanishing of m means that the infinitesimal body is coincident with E. The case where m vanishes will be excluded, and then m_0 cannot vanish. Therefore, if $P_2^{(3)}$ vanishes, for a sufficiently small value of $|m_0|$, it vanishes identically. The identity must hold with respect to the η of equation (6), and therefore the constant parts of the second and third terms of R_3 arising from the left members of (6) must be identically zero. By (34) and (36) it is seen that p_2 is a function of m as well as of m_0, and, by (40), z_1 contains no m. The identity must also hold with respect to m. Since $\chi_1^{(c)}$, $\chi_3^{(s)}$ contain m as a factor, an investigation of equations (55) and their solutions shows that p_2 contains terms of the first and second degrees in m, as well as terms independent of m, and that there are terms in m^2 with constant coefficients distinct from zero. Therefore the constant parts

of the last three terms of R_3 are not identically zero, and $P_2^{(3)}$ differs from zero for all sufficiently small values of $|m_0|$.

Then $c_1^{(1)}$ is determined by (64), except for sign, and when (64) is satisfied, the general solution of (61) is periodic and has the form

$$(65)\quad z_3 = c_1^{(3)} e^{\sigma\sqrt{-1}\,\tau}\,\xi^{(1)}(m_0;\tau) + c_2^{(3)} e^{-\sigma\sqrt{-1}\,\tau}\,\xi^{(2)}(m_0;\tau) + Q_1^{(3)}(\tau) + Q_3^{(3)}(\tau),$$

where $Q_1^{(3)}$ and $Q_3^{(3)}$ are homogeneous polynomials, of the first and third degrees respectively, in $c_1^{(1)} e^{\sigma\sqrt{-1}\,\tau}$ and $c_2^{(1)} e^{-\sigma\sqrt{-1}\,\tau}$. By the properties (b'') and (c'') of R_3 and by the forms of the $\xi^{(i)}$ given in (41), it follows from equations (61) and (62) that $Q_1^{(3)}$ and $Q_3^{(3)}$ have the fundamental properties (b'') and (c'') of R_3. Therefore $Q_1^{(3)}(0) = Q_3^{(3)}(0) = 0$, and since $z_3(0) = 0$, it follows that $c_1^{(3)}$ and $c_2^{(3)}$ are subject to the relation

$$(66)\quad c_1^{(3)} + c_2^{(3)} = 0.$$

This leaves one undetermined constant at this step.

The Fourth Step.—The coefficients of λ^2 are defined by the differential equations

$$\ddot{p}_4 + \theta_1^{(c)}(m_0;\tau)\,p_4 + \theta_4^{(s)}(m_0;\tau)\,u_4 + \theta_5^{(c)}(m_0;\tau)\,\dot{u}_4 = P_4,$$

$$r_0\ddot{u}_4 + \phi_1^{(s)}(m_0;\tau)\,p_4 + \phi_2^{(c)}(m_0;\tau)\,\dot{p}_4 + \phi_5^{(c)}(m_0;\tau)\,u_4 = Q_4.$$

By the properties (a), ..., (δ) of (19), and the fundamental properties of z_1, z_3, p_2, u_2, it follows that P_4 and Q_4 have the following properties:

(a') P_4 and Q_4 are polynomials in $c_1^{(1)} e^{\sigma\sqrt{-1}\,\tau}$, ..., $c_2^{(3)} e^{-\sigma\sqrt{-1}\,\tau}$, whose coefficients are sine and cosine series of integral multiples of τ and whose terms are all of even degree; the highest degree is the fourth, and there is no absolute term.

(b') P_4 and Q_4 are unchanged by interchanging $c_1^{(j)}$ and $c_2^{(j)}$ $(j = 1, 3)$, and changing the sign of $\sqrt{-1}$.

(c') In the coefficients of P_4, the coefficients of the cosine terms are real, and the coefficients of the sine terms are purely imaginary.

(d') In the coefficients of Q_4, the coefficients of the sine terms are real and the coefficients of the cosine terms are purely imaginary.

(e') In P_4 the term independent of the exponentials is $(c_1^{(1)})^2 (c_2^{(1)})^2$ multi-

plied by a cosine series, and in Q_4 the corresponding term is $(c_1^{(1)})^2 (c_2^{(1)})^2$ multiplied by a sine series.

On comparing equations (67) and the properties (a'), ..., (d') of P_4 and Q_4 with equations (53), and the properties (a'), ..., (d') of P_2 and Q_2, it is seen that the equations and fundamental properties are the same. Therefore the same argument as used in the second step will show that the solution of (67) has the properties (a'), ..., (d') of P_4 and Q_4, and that the arbitrary constants arising at this step are all determined by the conditions that the solution shall be symmetric and periodic. An investigation of the method also shows that the property (e') of P_4 and Q_4 is likewise a property of the solution.

The Fifth Step.—In order to make the induction to the general step it is necessary to carry the integration one step further. The differential equation defining the coefficient of λ^5 is

$$\ddot{z} + \psi^{(c)}(m_0\,;\,\tau)\, z_5 = 0. \tag{68}$$

By the properties (α), ..., (δ) of (19), R_5 is made up of terms in z_3, $p_2 z_3$, $p_4 z_1$, $p_2^2 z_1$, $u_2^2 z_1$, $z_1^2 z_3$, and z_1^5 each multiplied by cosine series, and of terms in $u_2 z_3$ and $u_4 z_1$ each multiplied by sine series. From these facts and the fundamental properties of z_1, ..., u_4, it follows that R_5 has the following properties:—

(a'') It is an odd degree polynomial of the fifth order in $c_1^{(1)} e^{\sigma\sqrt{-1}\,\tau}$, ..., $c_2^{(3)} e^{-\sigma\sqrt{-1}\,\tau}$, whose coefficients are sine and cosine series of integral multiples of τ.

(b'') It is unchanged by interchanging $c_1^{(j)}$ and $c_2^{(j)}$ $(j = 1, 3)$, and changing the sign of $\sqrt{-1}$.

(c'') In its coefficients, the coefficients of the cosine terms are real, and the coefficients of the sine terms are purely imaginary.

(d'') The constants $c_1^{(3)}$, $c_2^{(3)}$ appear only to the first degree.

The general solution of the left member of (68) set equal to zero is

$$z_5 = C_1^{(5)} e^{\sigma\sqrt{-1}\,\tau}\, \xi^{(1)}(m_0\,;\,\tau) + C_2^{(5)} e^{-\sigma\sqrt{-1}\,\tau}\, \xi^{(2)}(m_0\,;\,\tau), \tag{69}$$

where $C_1^{(5)}$, $C_2^{(5)}$ are arbitrary constants. In order that (69) shall be a

solution of the whole of (68), $C_1^{(5)}$ and $C_2^{(5)}$ must satisfy the differential equations

$$(70) \qquad \begin{cases} \Delta_2 \dot{C}_1^{(5)} = -\xi^{(2)} R_5 e^{-\sigma\sqrt{-1}\,\tau}, \\ \Delta_2 \dot{C}_2^{(5)} = \quad \xi^{(1)} R_5 e^{\sigma\sqrt{-1}\,\tau}, \end{cases}$$

where Δ_2 has the same meaning as before. In order that the solutions of (70) shall be periodic, the constant parts of their right members must be zero. From the character of the terms of which R_5 is made up, and from its properties (a''), ..., (d''), it follows that the conditions that these constant parts shall vanish are of the form

$$(71) \qquad \begin{cases} P_1^{(5)} c_1^{(3)} + P_2^{(5)} c_1^{(1)} c_2^{(1)} c_1^{(3)} + P_3^{(5)} (c_1^{(1)})^2 c_2^{(3)} + P_4^{(5)} (c_1^{(1)})^3 (c_2^{(1)})^2 = 0, \\ P_1^{(5)} c_2^{(3)} + P_2^{(5)} c_2^{(1)} c_1^{(1)} c_2^{(3)} + P_3^{(5)} (c_2^{(1)})^2 c_1^{(3)} + P_4^{(5)} (c_2^{(1)})^3 (c_1^{(1)})^2 = 0, \end{cases}$$

where $P_1^{(5)}$, ..., $P_4^{(5)}$ are power series in m_0.

Since $$c_1^{(1)} = -c_2^{(1)} = \pm\sqrt{P_1^{(3)}/P_2^{(3)}}, \qquad c_1^{(3)} = -c_2^{(3)},$$

equations (71) are equivalent to the single equation

$$(72) \qquad K_3 c_1^{(3)} + L_3 c_1^{(1)} = 0,$$

where K_3 and L_3 are power series in m_0. Since m_0 is distinct from zero, it follows that, if K_3 vanishes for a sufficiently small value of $|m_0|$, it must vanish identically. An investigation, similar to that made for equation (64), shows that the part of K_3 arising from the term in $p_2 z_3$ contains a term in m^2 alone with a constant coefficient different from zero, and that m^2 nowhere else occurs alone in K_3. Therefore K_3 does not vanish indentically, and hence it is distinct from zero when $|m_0|$ is sufficiently small. Then equation (62) determines $c_1^{(3)}$ uniquely and linearly in terms of $c_1^{(1)}$.

When $c_1^{(3)}$ has been determined by (72), the solution of (69) is periodic and has the form

(73)

$$z_5 = c_1^{(5)} e^{\sigma\sqrt{-1}\,\tau}\, \xi^{(1)}(m_0\,;\,\tau) + c_2^{(5)} e^{-\sigma\sqrt{-1}\,\tau}\, \xi^{(2)}(m_0\,;\,\tau) + Q_1^{(5)}(\tau) + Q_3^{(5)}(\tau) + Q_5^{(5)}(\tau),$$

where $Q_1^{(5)}$, $Q_3^{(5)}$, $Q_5^{(5)}$ are homogeneous polynomials, of the first, third, and fifth degrees respectively, in $c_1^{(1)} e^{\sigma\sqrt{-1}\,\tau}$, ..., $c_2^{(3)} e^{-\sigma\sqrt{-1}\,\tau}$. By the properties (b'') and (c'') of R_5, and by the properties of the $\xi^{(i)}$ given in (41), it follows

from equations (69) and (70) that $Q_1^{(5)}$, $Q_3^{(5)}$, and $Q_5^{(5)}$ have the properties (b'') and (c'') of R_5. Therefore

$$Q_1^{(5)}(0) = Q_3^{(5)}(0) = Q_5^{(5)}(0) = 0,$$

and, since $z_5(0) = 0$, it follows that the arbitrary constants $c_1^{(5)}$ and $c_2^{(5)}$ must satisfy the relation

$$c_1^{(5)} + c_2^{(5)} = 0. \tag{74}$$

9. *The Inductions to the General Steps in the Solution of* (19).

The inductions to the general steps in the solution will now be made. Suppose that the $z_1, p_2, u_2, \ldots, p_{2s-2}, u_{2s-2}, z_{2s-1}$ have been computed, and that they have the following properties :—

(a) The p_{2j} and u_{2j} are even degree polynomials in $c_1^{(2k-1)} e^{\sigma\sqrt{-1}\,\tau}$ and $c_2^{(2k-1)} e^{-\sigma\sqrt{-1}\,\tau}$ ($k = 1, \ldots, j$), whose coefficients are sine and cosine series of integral multiples of τ; the order of the polynomials is $2j$, and there are no absolute terms.

(b) The z_{2j-1} are odd degree polynomials in $c_1^{(2k-1)} e^{\sigma\sqrt{-1}\,\tau}$ and $c_2^{(2k-1)} e^{-\sigma\sqrt{-1}\,\tau}$ ($k = 1, \ldots, j-1$), whose coefficients are sine and cosine series of integral multiples of τ; the order of the polynomials being $2j-1$.

(c) The p_{2j}, u_{2j}, z_{2j-1} are unchanged by interchanging $c_1^{(2k-1)}$ and $c_2^{(2k-1)}$ ($k = 1, \ldots, s$), and changing the sign of $\sqrt{-1}$.

(d) In the coefficients of p_{2j} and z_{2j-1}, the coefficients of the cosine terms are real, and the coefficients of the sine terms are purely imaginary.

(e) In the coefficients of u_{2j}, the coefficients of the sine terms are real, and the coefficients of the cosine terms are purely imaginary.

(f) The arbitrary constants arising at the $2j$-th steps have all been determined to satisfy the symmetry and periodicity conditions; the constants arising at the $(2j-1)$-th steps satisfy the relation

$$c_1^{(2j-1)} + c_2^{(2j-1)} = 0,$$

and have all been determined uniquely and linearly in terms of $c_1^{(1)}$ by the periodicity conditions, except

$$c_1^{(2s-1)} = -c_2^{(2s-1)}.$$

Thus there is one arbitrary constant.

The 2s-th Step.—The differential equations defining the coefficients of λ^s are

$$(75)\quad \begin{cases} \ddot{p}_{2s}+\theta_1^{(c)}(m_0\,;\,\tau)\,p_{2s}+\theta_4^{(s)}(m_0\,;\,\tau)\,u_{2s}+\theta_5^{(c)}(m_0\,;\,\tau)\,\dot{u}_{2s} = P_{2s}, \\ r_0\,\ddot{u}_{2s}+\phi_1^{(s)}(m_0\,;\,\tau)\,p_{2s}+\phi_2^{(c)}(m_0\,;\,\tau)\,\dot{p}_{2s}+\phi_5^{(s)}(m_0\,;\,\tau)\,u_{2s} = Q_{2s}. \end{cases}$$

By properties (a), ..., (δ) of (19), P_{2s} and Q_{2s} consist of terms of the form

$$G_{2s}(\tau)\,p_2^{\lambda_2}\ldots p_{2i}^{\lambda_{2i}}u_2^{\mu_2}\ldots u_{2j}^{\mu_{2j}}z_1^{\nu_1}\ldots z_{2k-1}^{\nu_{2k-1}},$$

where $\nu_1+\ldots+\nu_{2k-1}$ is an even integer; in P_{2s}, according as $\mu_2+\ldots+\mu_{2j}$ is even or odd, $G_{2s}(\tau)$ is a cosine or a sine series respectively of integral multiples of τ; in Q_{2s}, according as $\mu_2+\ldots+\mu_{2j}$ is odd or even, $G_{2s}(\tau)$ is a cosine or a sine series; by properties (a) and (b), the inequality

$$2\lambda_2+\ldots+2i\lambda_{2i}+2\mu_2+\ldots+2j\mu_{2j}+\nu_1+\ldots+(2k-1)\,\nu_{2k-1} \leqslant 2s$$

must be satisfied. It follows from these considerations, and the properties (a), ..., (e) of p_{2j}, u_{2j}, z_{2j-1}, that P_{2s} and Q_{2s} have the following properties:—

(a') P_{2s} and Q_{2s} are even degree polynomials in $c_1^{(2k-1)}e^{\sigma\sqrt{-1}\,\tau}$ and $c_2^{(2k-1)}e^{-\sigma\sqrt{-1}\,\tau}$ $(k = 1, \ldots, s)$, whose coefficients are sine and cosine series of integral multiples of τ; the highest degrees are $2s$, and there are no absolute terms.

(b') P_{2s} and Q_{2s} are unchanged by interchanging $c_1^{(2k-1}$ and $c_2^{(2k-1)}$ $(k = 1, \ldots, s)$ and changing the sign of $\sqrt{-1}$.

(c') In the coefficients of P_{2s}, the coefficients of cosine terms are real, and the coefficients of the sine terms are purely imaginary.

(d') In the coefficients of Q_{2s}, the coefficients of the sine terms are real, and the coefficients of the cosine terms are purely imaginary.

(e') In P_{2s} and Q_{2s}, $c_1^{(2s-1)}$ and $c_2^{(2s-1)}$ enter linearly.

On comparing equations (75) with equations (53), it is seen that they are of the same form and that the fundamental properties of P_{2s} and Q_{2s} are of the same nature as the fundamental properties of P_2 and Q_2. Then, by a discussion similar to the discussion of the solution of (53), it can be shown that the fundamental properties (a'), ..., (e') of P_{2s} and Q_{2s} are also properties of p_{2s} and u_{2s} respectively, and that the constants of integration arising at this step are determined by the symmetry and periodicity conditions. Therefore the properties (a), ..., (f), so far as they pertain to p_{2j} and u_{2j}, also hold for $j = s$.

The $(2s+1)$*-th Step.*—The coefficient of $\lambda^{\frac{1}{2}(2s+1)}$ is defined by the differential equation

$$\ddot{z}_{2s+1}+\psi^{(c)}(m_0\,;\,\tau)\,z_{2s+1} = R_{2s+1}. \tag{76}$$

By properties (a), ..., (δ) of (19), R_{2s+1} consists of terms of the form

$$G_{2s+1}(\tau)\,p_2^{\lambda_2}\ldots p_{2i}^{\lambda_{2i}}\,u_2^{\mu_2}\ldots u_{2j}^{\mu_{2j}}z_1^{\nu_1}\ldots z_{2k-1}^{\nu_{2k-1}},$$

where $\nu_1+\ldots+\nu_{2k-1}$ is an odd integer; $G_{2s+1}(\tau)$ is a cosine or a sine series of integral multiples of τ according as $\mu_2+\ldots+\mu_{2j}$ is even or odd; by properties (a) and (b), the inequality

$$2\lambda_2+\ldots+2i\lambda_{2i}+2\mu_2+\ldots+2j\mu_{2j}+\nu_1+\ldots+(2k-1)\,\nu_{2k-1} \leqslant 2s+1$$

must be satisfied. From these properties of the terms of R_{2s+1}, and from the properties (a), ..., (e) of z_1, p_2, u_2, ..., z_{2s-1}, p_{2s}, u_{2s}, it follows that R_{2s+1} has the following properties:—

(a'') It is an odd degree polynomial, of order $2s+1$, in $c_1^{(2k-1)}e^{\sigma\sqrt{-1}\,\tau}$ and $c_2^{(2k-1)}e^{-\sigma\sqrt{-1}\,\tau}$ $(k = 1, \ldots, s)$, whose coefficients are sine and cosine series of integral multiples of τ.

(b'') It is unchanged by interchanging $c_1^{(2k-1)}$ and $c_2^{(2k-1)}$ $(k = 1, \ldots, s)$, and changing the sign of $\sqrt{-1}$.

(c'') In its coefficients, the coefficients of the cosine terms are real, and the coefficients of the sine terms are purely imaginary.

(d'') The constants $c_1^{(2s-1)}$ and $c_2^{(2s-1)}$ occur only to the first degree.

The general solution of the left members of (76) set equal to zero is

$$z_{2s+1} = C_1^{(2s+1)}e^{\sigma\sqrt{-1}\,\tau}\,\xi^{(1)}(m_0\,;\,\tau)+C_2^{(2s+1)}e^{-\sigma\sqrt{-1}\,\tau}\,\xi^{(2)}(m_0\,;\,\tau), \tag{77}$$

where $C_1^{(2s+1)}$, $C_2^{(2s+1)}$ are arbitrary constants. In order that (77) shall be a solution of the complete equation (76), $C_1^{(2s+1)}$ and $C_2^{(2s+1)}$ must satisfy the differential equations

$$(78)\qquad \begin{cases} \Delta_2 \dot{C}_1^{(2s+1)} = -\xi^{(2)} R_{2s+1} e^{-\sigma\sqrt{-1}\,\tau}, \\ \Delta_2 \dot{C}_2^{(2s+1)} = \quad \xi^{(1)} R_{2s+1} e^{\sigma\sqrt{-1}\,\tau}, \end{cases}$$

and in order that the solutions of (78) shall be periodic, the constant parts of the right hand members must be equal to zero. By properties $(a), \ldots, (e)$ of the p_{2j}, u_{2j}, z_{2j-1} $(j = 1, \ldots, s)$, the properties (a''), ..., (d'') of R_{2s+1}, and the properties of $\xi^{(i)}$ given in (41), it follows that the conditions that these constant parts shall be zero are of the form

$$(79)\qquad \begin{cases} P_1^{(2s+1)} c^{(2s-1)} + P_2^{(2s+1)} c_2^{(2s-1)} + P_3^{(2s+1)} (c_1^{(2j+1)}, c_2^{(2k-1)}) = 0, \\ P_1^{(2s+1)} c_2^{(2s-1)} + P^{(2s+1)} c_1^{(2s-1)} + P_3^{(2s+1)} (c_2^{(2j+1)}, c_1^{(2k-1)}) = 0, \end{cases}$$

where $P_3^{(2s+1)}$ is an odd degree polynomial in

$$c_1^{(2j+1)},\ c_2^{(2k-1)} \quad (j, k = 1, \ldots, s-1);$$

and by properties (a'') and (d'') of R_{2s+1}, $P_1^{(2s+1)}$ and $P_2^{(2s+1)}$ have the forms

$$P_1^{(2s+1)} = P_1^{(5)} + P_2^{(5)} c_1^{(1)} c_2^{(1)},$$

$$P_2^{(2s+1)} = P_2^{(5)} (c_1^{(1)})^2.$$

Since $$c_1^{(2j-1)} = -c_2^{(2j-1)} \quad (j = 1, \ldots, s),$$

the two conditions (79) are equivalent, and by (71) and (72) they can be represented by the single equation

$$(80)\qquad CK_3 c_1^{(2s-1)} + L_{2s+1} c_1^{(1)},$$

where C is a constant different from zero, and L_{2s+1} is a power series in m_0. Since, as we saw before, K_3 is different from zero when $|m_0|$ is sufficiently small and distinct from zero, equation (80) determines c_1^{2s-1} uniquely and linearly in terms of $c_1^{(1)}$.

When equation (80) has been satisfied, the solutions of (78) are periodic, and the general solution of (77) is of the form

$$(81)\qquad z_{2s+1} = c_1^{(2s+1)} e^{\sigma\sqrt{-1}\,\tau} \xi^{(1)}(m_0\,;\ \tau) + c_2^{(2s+1)} e^{-\sigma\sqrt{-1}\,\tau} \xi^{(2)}(m_0\,;\ \tau) + \sum_{j=0}^{s} Q_{2j+1}^{(2s+1)}(\tau),$$

where the $Q_{2j+1}^{(2s+1)}$ are homogeneous polynomials of degree $2j+1$ in

$$c_1^{(2k-1)} e^{\sigma\sqrt{-1}\,\tau} \text{ and } c_2^{(2k-1)} e^{-\sigma\sqrt{-1}\,\tau} \quad (k = 1, \ldots, s).$$

By the properties (b'') and (c'') of R_{2s+1}, and the forms of $\xi^{(i)}$ given in (41), it follows that the $Q_{2j+1}^{(2s+1)}$ have the properties (b'') and (c'') of R_{2s+1}; and since

$$c_1^{(2k-1)} = -c_2^{(2k-1)} \quad (k = 1, \ldots, s),$$

it follows that

$$Q_{2j+1}^{(2s+1)}(0) = 0 \quad (j = 0, \ldots, s).$$

By the initial conditions $z_{2s+1}(0) = 0$, and consequently the arbitrary constants $c_1^{(2s+1)}$, $c_2^{(2s+1)}$ must satisfy the relation

$$c_1^{(2s+1)} + c_2^{(2s+1)} = 0. \tag{82}$$

Therefore one constant remains undetermined, and z_{2s+1} has the properties (a), ..., (f) so far as they pertain to z_{2j-1}.

For particular values of s, however, constant terms can arise from the right hand members of (78), other than those which arise in the manner considered. There are terms of the type

$$(c_1^{(1)})^{2s+1} e^{(2s+1)\sigma\sqrt{-1}\,\tau} \{R^{(c)}(\tau) + \sqrt{-1}\, R^{(s)}(\tau)\},$$

$$(c_2^{(1)})^{2s+1} e^{-(2s+1)\sigma\sqrt{-1}\,\tau} \{R^{(c)}(\tau) - \sqrt{-1}\, R^{(s)}(\tau)\},$$

which can be written in the form

$$(c_1^{(1)})^{2s+1} e^{\sigma\sqrt{-1}\,\tau} (\cos 2s\sigma\tau + \sqrt{-1} \sin 2s\sigma\tau) \{R^{(c)}(\tau) + \sqrt{-1}\, R^{(s)}(\tau)\},$$

$$(c_2^{(1)})^{2s+1} e^{-\sigma\sqrt{-1}\,\tau} (\cos 2s\sigma\tau - \sqrt{-1} \sin 2s\sigma\tau) \{R^{(c)}(\tau) - \sqrt{-1}\, R^{(s)}(\tau)\}.$$

Since $\sigma = N/\nu$, where N and ν are relatively prime integers, constants can arise from terms of this type when s is equal to any multiple of ν, but it is readily seen that these terms do not change the fundamental character of equations (79).

It is also worth while to consider the divisors which arise at the successive steps of the integration. At the general step of the integration of the first two equations of (19) there arise quadratures of the form

$$\int e^{\pm(2k\sigma \pm \omega)\sqrt{-1}\,\tau} \begin{matrix}\cos \\ \sin\end{matrix} j\tau\, d\tau \quad (k = 0, \ldots, s),$$

and therefore the divisors

$$j^2 - \omega^2, \quad j^2 - (2\sigma \pm \omega)^2, \quad \ldots, \quad j^2 - (2s\sigma \pm \omega)^2$$

are introduced. At the general step of the solution of the third equation of (19) there arise quadratures of the form

$$\int e^{\pm(2k+1)\sigma\sqrt{-1}\,\tau} \begin{matrix}\cos\\ \sin\end{matrix} j\tau\, d\tau \quad (k = 0, \ldots, s),$$

and the corresponding divisors are

$$j^2-\sigma^2, \quad j^2-9\sigma^2, \quad \ldots, \quad j^2-(2s+1)^2\sigma^2.$$

The divisors $j^2-(2k\sigma\pm\omega)^2$ can vanish only when $2k\sigma\pm\omega$ is an integer. But

$$2k\sigma\pm\omega = 2k\pm1+\sum_{j=1}^{\infty}(2k\sigma_j\pm\omega_j)\,m_0^j,$$

and therefore $2k\sigma\pm\omega$ is an integer, for $|m_0|$ sufficiently small, only when

$$2k\sigma_j\pm\omega_j \equiv 0 \quad (j = 1, \ldots, \infty).$$

But this identity cannot hold, since the fundamental equation for ω contains a explicitly while that for σ does not.

The divisors $j^2-(2k+1)^2\sigma^2$ will vanish, if ν is odd, for $2k+1 = \nu$ and $j = N$. But when this occurs the integrals are of the type

$$\int e^{h\sqrt{-1}\,\tau} \begin{matrix}\cos\\ \sin\end{matrix} h\tau\, d\tau = \underset{+\sqrt{-1}}{+}\left(\frac{e^{2h\sqrt{-1}\,\tau}}{4h\sqrt{-1}}\right) \underset{-\sqrt{-1}}{+}\left(\frac{\tau}{2}\right),$$

where h is an integer. The non-periodic terms arising from these integrations are eliminated along with the other non-periodic terms at the same step.

16. *Conclusion.*

The possibility of constructing a symmetric periodic solution of the differential equations (19) as power series in the parameter $\lambda^{\frac{1}{3}}$ has been shown, and when the solution thus obtained is substituted in equations (9) and (18), the results give the semi-polar coordinates r, θ, z of the infinitesimal body. In §§ 8, 9 the construction of the solution was shown to be unique when the sign of $c_1^{(1)}$ is chosen, and from the properties (a), ..., (f) of the solution of (19) it follows that r and θ are even functions of $c_1^{(1)}$, while z is an odd function of the same quantity. Further, r and θ are power series in integral powers of λ, while z is a power series in odd powers of $\lambda^{\frac{1}{3}}$. Therefore r and θ are unique, while z is determined except for sign. This means physically, as was stated before, that at the

initial time the infinitesimal body is at a certain apse, but that it may be at either node; and from the form of the initial conditions (10) in Prof. Moulton's memoir, it follows that it is at its apse in conjunction.

The solution for the case when the infinitesimal body starts at its apse in opposition is obtained by writing $\tau+\pi$ for τ in the generating solution, which is equivalent to changing the sign of M_1. And since this is the only way in which M_1 enters the discussion, the solution will differ from that just obtained only in the sign of M_1. Hence for given values of N and ν, we get four solutions by taking $\pm M_1$, $\pm\lambda^{\frac{1}{2}}$. There are three possible cases* :—

(i) N and ν both odd; then if the infinitesimal body starts at the $\left.\begin{matrix}\text{ascending}\\\text{descending}\end{matrix}\right\}$ node and in $\left.\begin{matrix}\text{conjunction}\\\text{opposition}\end{matrix}\right\}$, at the half period it will be at the $\left.\begin{matrix}\text{descending}\\\text{ascending}\end{matrix}\right\}$ node and in $\left.\begin{matrix}\text{opposition}\\\text{conjunction}\end{matrix}\right\}$.

(ii) N odd and ν even; then if the infinitesimal body starts at the $\left.\begin{matrix}\text{ascending}\\\text{descending}\end{matrix}\right\}$ node and in $\left.\begin{matrix}\text{conjunction}\\\text{opposition}\end{matrix}\right\}$, at the half period it will be at the $\left.\begin{matrix}\text{descending}\\\text{ascending}\end{matrix}\right\}$ node and in $\left.\begin{matrix}\text{conjunction}\\\text{opposition}\end{matrix}\right\}$.

(iii) N even and ν odd; then if the infinitesimal body starts at the $\left.\begin{matrix}\text{ascending}\\\text{descending}\end{matrix}\right\}$ node and in $\left.\begin{matrix}\text{conjunction}\\\text{opposition}\end{matrix}\right\}$, at the half period it will be at the $\left.\begin{matrix}\text{ascending}\\\text{descending}\end{matrix}\right\}$ node and in $\left.\begin{matrix}\text{opposition}\\\text{conjunction}\end{matrix}\right\}$.

Therefore in all three cases there are precisely two geometrically distinct orbits; a pair of solutions belongs to each orbit, the positions of the infinitesimal body for the two solutions being separated by half a period.

When the generalization (13) of m was made in equations (10), it was specified that the m which appeared implicitly in τ was not generalized. This was necessary in order that the synodic period given by the solution

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* To get four solutions from (i) read thus:

... ascending ... conjunction, ... descending ... opposition,
... descending ... opposition, ... ascending ... conjunction,
... ascending ... opposition, ... descending ... conjunction,
... descending ... conjunction, ... ascending ... opposition.

Similarly for (ii) and (iii).

should still be the observed synodic period of the three bodies ; viz., 2π in τ or $2.m/n'.\pi$ in t. In order to make use of the solutions of the equations of variation which have been found, it is necessary to generalize the m where it appears elsewhere in the left members of (10), but the utmost freedom is permissible in the right members. For example, the m does not need to be generalized at all in the right members, or it might be generalized only in the m which appears as a factor of each right member. These are only two of an infinite number of possibilities. The question of the exact mode of generalization to be adopted will be left open until the actual construction of the solution is considered, since it may then appear that some particular mode of generalization has advantages.

To complete the problem practically it remains to construct the particular solutions (22) of (21), and then to compute the solution of equation (19) as far as may be desired.

THE GREEN'S FUNCTION FOR THE EQUATION $\nabla^2 u + \kappa^2 u = 0$

(SECOND PAPER)

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Introduction.

In his recent paper on "A Class of Diffraction Problems,"* Macdonald deduces the solution for a source of sound between two planes cutting at any angle from the corresponding solution of the equation of potential; and he puts in analytical form his argument for the general validity of this method.

In this paper I desire to call attention to the fact that, with the same assumptions as are involved in the usual treatment of the differential equations of mathematical physics, the Green's functions, both for the equation $\nabla^2 u = 0$ and the equation $\nabla^2 u + \kappa^2 u = 0$, can be obtained immediately from an integral equation, when the region with which we are dealing is finite. The solution is as easy for the one equation as for the other in the case when the region is bounded by one sphere (or two concentric spheres), a coaxial right circular cone (or two such cones), and two axial planes; and again when the boundary consists of a circular cylinder (or two coaxial circular cylinders), two axial planes, and two planes perpendicular to the axis.

Also Fourier's method leads as readily to the solution of the equation $\nabla^2 u + \kappa^2 u = 0$ as to that of the equation $\nabla^2 u = 0$. This method has been applied by Sommerfeld to the Green's function for the former equation.†

For infinite regions neither Fourier's method nor that based upon the integral equation can be applied to the equation $\nabla^2 u + \kappa^2 u = 0$. For the potential problem we can, however, proceed from the finite to the infinite

* *Proc. London Math. Soc.*, Ser. 2, Vol. 14, p. 410 (1915).

† *Jahresbericht der Deutschen Mathematiker-Vereinigung*, Bd. 21, p. 309 (1913).

region by supposing the outer boundary to move off to infinity. For this reason Macdonald's method is of special value in this large class of diffraction problems.

In my previous paper with the same title* a number of these problems were solved with the aid of contour integrals, without the assumptions above referred to. Similar rigorous proofs for a larger class of potential problems were given some years ago by Dougall.†

1. It is well known that the Green's function‡ for the equation $\nabla^2 u = 0$ appears as the kernel of a homogeneous integral equation.§

Let $K(x, y, z; x', y', z')$ be the Green's function for this equation in the region bounded by the closed surface S.

Let $\psi(x, y, z)$ be any solution of the equation

$$\nabla^2 \psi + \lambda \psi = 0, \tag{1}$$

which vanishes on the surface of S, and, with its first and second differential coefficients, is finite and continuous in S.

Applying Green's theorem to the region between S and a small sphere Σ, whose centre is at (x', y', z'), we have

$$\iiint (\psi \nabla^2 K - K \nabla^2 \psi)\, dx\, dy\, dz + \iint \left(\psi \frac{\partial K}{\partial n} - K \frac{\partial \psi}{\partial n} \right) d\Sigma = 0, \tag{2}$$

where $\partial / \partial n$ denotes differentiation along the outward-drawn normal to Σ.

Since the Green's function is infinite as $1/4\pi r$, when $r \to 0$, at the point (x', y', z'), if we suppose the radius of Σ to tend to zero, we obtain, from (2),

$$\psi(x', y', z') = \lambda \iiint K(x, y, z; x', y', z')\, \psi(x, y, z)\, dx\, dy\, dz, \tag{3}$$

the integration being taken through S.

A simple application of Green's theorem shows that $K(x, y, z; x', y', z')$

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* *Proc. London Math. Soc.*, Ser. 2, Vol. 13, p. 236 (1914).

† *Proc. Edinburgh Math. Soc.*, Vol. 18, p. 33 (1900).

‡ In this paper I take the infinity of the Green's function at (x', y', z') to be as $1/4\pi r$ in the potential problem (and $e^{-ikr}/4\pi r$ in the wave problem). In comparing results, the factor $1/4\pi$ has to be noted. Also I take the surface condition at S to be $u = 0$. The cases $\partial u/\partial n = 0$, $\partial u/\partial n + hu = 0$ admit of the same treatment.

§ Cf. Kneser, *Die Integralgleichungen und ihre Anwendungen in der Mathematischen Physik* (Braunschweig, 1911); Heywood-Fréchet, *L'Équation de Fredholm et ses applications à la physique mathématique* (Paris, 1912).

is a symmetrical function of (x, y, z) and (x', y', z'); in other words, we have

$$K(x, y, z\,;\ x', y', z') = K(x', y', z'\,;\ x, y, z).$$

It follows, from (3), that the functions ψ are characteristic functions (*Eigenfunctionen*), and the numbers λ are characteristic numbers (*Eigenwerte*) of a homogeneous integral equation with $K(x, y, z\,;\ x', y', z')$ as its symmetrical kernel.

2. Denoting the characteristic numbers by

$$\lambda_1,\ \lambda_2,\ \ldots,$$

and the corresponding characteristic functions by

$$\psi_1,\ \psi_2,\ \ldots,$$

from Green's theorem we have at once

$$\iiint \psi_r \psi_s\, dx\, dy\, dz = 0 \quad (r \neq s), \tag{4}$$

the integration being taken through S.

And we suppose the functions normalised, so that

$$\iiint \psi^2\, dx\, dy\, dz = 1\ . \tag{5}$$

If we assume that the Green's function can be expanded in the series

$$A_1\psi_1 + A_2\psi_2 + \ldots,$$

and that the series can be integrated term by term, we have, from (4) and (5),

$$A_r = \iiint K(x, y, z\,;\ x', y', z')\, \psi_r(x, y, z)\, dx\, dy\, dz$$

$$= \frac{\psi_r(x', y', z')}{\lambda_r},$$

from (3). Thus we have, subject to these assumptions,*

$$K(x, y, z\,;\ x', y', z') = \sum_r \frac{\psi_r(x, y, z)\, \psi_r(x', y', z')}{\lambda_r}. \tag{6}$$

3. The assumptions as to the possibility of expressing the Green's

* If it can be proved that the series on the right of (6) is uniformly convergent, a fundamental theorem in the theory of integral equations allows us to equate $K(x, y, z\,;\ x', y', z')$ to the sum of the series.

function by the series of characteristic functions are practically the same as those which are made in the usual treatment of the potential problems.* It follows that we need not hesitate to use (6) for such problems, when the functions ψ and the corresponding numbers λ can be obtained. In my paper "On Integral Equations and the Determination of Green's Functions in the Theory of Potential,"† the solutions are given for all the cases mentioned in the introduction.

4. The following examples, solved by this method, will be referred to below :—

I. *The region bounded by the sphere* $r = a$, *the cone* $\theta = \theta_0$, *and the axial planes* $\phi = 0$, $\phi = \alpha$.

Here we have

$$\psi_{\rho n m} = \sqrt{\left(\frac{2}{\alpha r}\right)} \frac{J_{n+\frac{1}{2}}(\rho r)\, P_n^{-m\pi/\alpha}(\mu) \sin \frac{m\pi}{\alpha}\phi}{\left[\int_0^a r J_{n+\frac{1}{2}}^2(\rho r)\, dr \int_{\mu_0}^1 \left[P_n^{-m\pi/\alpha}(\mu)\right]^2 d\mu\right]^{\frac{1}{2}}},$$

where m is any positive integer,

n is any positive root of $P_n^{-m\pi/\alpha}(\mu_0) = 0$,

ρ is any positive root of $J_{n+\frac{1}{2}}(\rho a) = 0$,

and $$\lambda = \rho^2.$$

It follows, from (6), that the Green's function $K(r, \theta, \phi;\ r', \theta', \phi')$ is given by‡

$$\frac{2}{a\sqrt{(rr')}} \sum_\rho \sum_n \sum_m \frac{J_{n+\frac{1}{2}}(\rho r)\, J_{n+\frac{1}{2}}(\rho r')\, P_n^{-m\pi/\alpha}(\mu)\, P_n^{-m\pi/\alpha}(\mu') \sin \frac{m\pi}{\alpha}\phi \sin \frac{m\pi}{\alpha}\phi'}{\rho^2 \int_0^a r J_{n+\frac{1}{2}}^2(\rho r)\, dr \int_{\mu_0}^1 \left[P_n^{-m\pi/\alpha}(\mu)\right]^2 d\mu}. \quad (7)$$

II. *The region bounded by the spheres* $r = a$, $r = b$, *the cone* $\theta = \theta_0$, *and the axial planes* $\phi = 0$, $\phi = \alpha$.

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* Cf. Macdonald, *Proc. London Math. Soc.*, Ser. 1, Vol. XXVI, p. 156 (1895).

† *Proc. Edinburgh Math. Soc.*, Vol. 31, p. 71 (1913).

‡ *Loc. cit.*, p. 86. The Green's function for the wedge, which Macdonald obtains in his last paper, will be found here under Corollary (ii), deduced from the above by making $a \to \infty$ and $\mu_0 \to -1$.

Here we have

$$\psi_{\rho nm} = \sqrt{\left(\frac{2}{ar}\right)} \frac{U_{n+\frac{1}{2}}(\rho r)\, P_n^{-m\pi/\alpha}(\mu) \sin\frac{m\pi}{\alpha}\phi}{\left[\int_a^b r U_{n+\frac{1}{2}}^2(\rho r)\, dr \int_{\mu_0}^1 \left[P_n^{-m\pi/\alpha}(\mu)\right]^2 d\mu\right]^{\frac{1}{2}}},$$

where m is any positive integer,

n is any positive root of $P_n^{-m\pi/\alpha}(\mu_0) = 0$,

$U_n(x) = J_n(x)\, K_n(ib) - J_n(b)\, K_n(ix)$,*

ρ is any positive root of $U_n(\rho a) = 0$,

and $$\lambda = \rho^2.$$

It follows, from (6), that the Green's function is given by

$$\frac{2}{a\sqrt{(rr')}} \sum_\rho \sum_n \sum_m \frac{U_{n+\frac{1}{2}}(\rho r)\, U_{n+\frac{1}{2}}(\rho r')\, P_n^{-m\pi/\alpha}(\mu)\, P_n^{-m\pi/\alpha}(\mu') \sin\frac{m\pi}{\alpha}\phi \sin\frac{m\pi}{\alpha}\phi'}{\rho^2 \int_a^b r U_{n+\frac{1}{2}}^2(\rho r)\, dr \int_{\mu_0}^1 \left[P_n^{-m\pi/\alpha}(\mu)\right]^2 d\mu}. \quad (8)$$

5. In my paper with the same title,† I pointed out that the equation $\nabla^2 u + \kappa^2 u = 0$ can be treated in a similar way. But I did not enter into details, as my object in that paper was to obtain the Green's functions by a rigorous method; that is, without the assumptions involved in the treatment given in the preceding sections.

If $K(x, y, z; x', y', z')$ is the solution of the equation $\nabla^2 u + \kappa^2 u = 0$ which vanishes at the surface S, and is finite and continuous, as also its first and second differential coefficients, inside S, except at (x', y', z'), where it becomes infinite as $e^{-i\kappa r}/4\pi r$, where $r \to 0$, then $K(x, y, z; x', y', z')$ is the symmetrical kernel of a homogeneous integral equation.

Indeed, if $\psi(x, y, z)$‡ is a solution of the equation

$$\nabla^2 \psi + (\lambda + \kappa^2)\psi = 0, \quad (9)$$

which vanishes at the surface of S, and is finite and continuous, as also

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* $K_n(ix)$ as usual stands for $\frac{\pi}{2 \sin n\pi} e^{-\frac{1}{2}ni\pi}\left[J_{-n}(x) - e^{ni\pi} J_n(x)\right]$.

† *Loc. cit.*, p. 237.

‡ The value $\lambda = 0$ is excluded, since we assume that κ^2 does not belong to one of the normal modes of vibration.

its first and second differential coefficients, inside S, we have

$$\psi(x', y', z') = \lambda \iiint K(x, y, z\,;\ x', y', z')\, \psi(x, y, z)\, dx\, dy\, dz, \qquad (10)$$

the integration being taken through S.

And with the same notation as in § 2, we have

$$\iiint \psi_r \psi_s\, dx\, dy\, dz = 0 \quad (r \neq s),$$

$$\iiint \psi_r^2\, dx\, dy\, dz \quad = 1.$$

Also, if we assume that $K(x, y, z\,;\ x', y', z')$ can be expanded in the series

$$A_1\psi_1 + A_2\psi_2 + \ldots,$$

and that the series can be integrated term by term, we have

$$A_r = \iiint K(x, y, z\,;\ x', y', z')\, \psi_r(x, y, z)\, dx\, dy\, dz$$

$$= \frac{\psi_r(x', y', z')}{\lambda_r}.$$

Thus we have, as before,

$$K(x, y, z\,;\ x', y', z') = \sum_r \frac{\psi_r(x, y, z)\, \psi_r(x', y', z')}{\lambda_r}. \qquad (11)$$

6. From (11) the Green's functions for the cases treated in § 4 (and other similar cases) can be written down :—

I. *The region bounded by the sphere $r = a$, the cone $\theta = \theta_0$, and the axial planes $\phi = 0$, $\phi = \alpha$.*

The Green's function* is given by

$$\frac{2}{\alpha\sqrt{(rr')}} \sum_\rho \sum_n \sum_m \frac{J_{n+\frac{1}{2}}(\rho r)\, J_{n+\frac{1}{2}}(\rho r')\, P_n^{-m\pi/\alpha}(\mu)\, P_n^{-m\pi/\alpha}(\mu') \sin\frac{m\pi}{\alpha}\phi \sin\frac{m\pi}{\alpha}\phi'}{(\rho^2 - \kappa^2)\int_0^a r J_{n+\frac{1}{2}}^2(\rho r)\, dr \int_{\mu_0}^1 [P_n^{-m\pi/\alpha}(\mu)]^2\, d\mu}, \qquad (12)$$

with the same notation as in § 4, I.

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* The solution for the region bounded by the sphere, and the axial planes, is obtained by letting $\mu_0 \to -1$, as in my paper on potential (*loc. cit.*, §§ 11, 13).

II. *The region bounded by the spheres* $r = a$, $r = b$, *the cone* $\theta = \theta_0$, *and the axial planes* $\phi = 0$, $\phi = \alpha$.

The Green's function is given by

$$\frac{2}{\alpha\sqrt{(rr')}} \sum_{\rho} \sum_{n} \sum_{m} \frac{U_{n+\frac{1}{2}}(\rho r)\, U_{n+\frac{1}{2}}(\rho r')\, P_n^{-m\pi/\alpha}(\mu)\, P_n^{-m\pi/\alpha}(\mu') \sin\frac{m\pi}{\alpha}\phi \sin\frac{m\pi}{\alpha}\phi'}{(\rho^2-\kappa^2)\int_a^b r U^2_{n+\frac{1}{2}}(\rho r)\, dr \int_{\mu_0}^1 [P_n^{-m\pi/\alpha}(\mu)]^2\, d\mu}, \quad (13)$$

with the same notation as in § 4, II.

7. The results of §§ 4, 6 can be simplified with the aid of the following theorems on Bessel's functions :—

I. *Let* n *be a given positive number, and* $\rho_1, \rho_2, \ldots$ *the positive roots of the equation*

$$J_n(\rho a) = 0.$$

Then we have

$$\sum_{\rho} \frac{J_n(\rho r)\, J_n(\rho r')}{\rho^2 \int_0^a r J_r^2(\rho r)\, dr} = \frac{1}{2n}\left(\frac{r}{a}\right)^n \left\{\left(\frac{a}{r'}\right)^n - \left(\frac{r'}{a}\right)^n\right\} \quad (0 < r < r')$$

$$= \frac{1}{2n}\left(\frac{r'}{a}\right)^n \left\{\left(\frac{a}{r}\right)^n - \left(\frac{r}{a}\right)^n\right\} \quad (r' < r < a). \quad (14)$$

II. *Let* n *be a given positive number, and* $\rho_1, \rho_2, \ldots$ *the positive roots of the equation*

$$U_n(\rho a) = J_n(\rho a)\, K_n(i\rho b) - J_n(\rho b)\, K_n(i\rho a) = 0 \quad (0 < a < b).$$

Then we have

$$\sum_{\rho} \frac{U_n(\rho r)\, U_n(\rho r')}{\rho^2 \int_a^b r U_n^2(\rho r)\, dr} = \frac{\frac{1}{2n}\left\{\left(\frac{r}{a}\right)^n - \left(\frac{a}{r}\right)^n\right\}\left\{\left(\frac{b}{r'}\right)^n - \left(\frac{r'}{b}\right)^n\right\}}{\left(\frac{b}{a}\right)^n - \left(\frac{a}{b}\right)^n} \quad (a < r < r')$$

$$= \frac{\frac{1}{2n}\left\{\left(\frac{r'}{a}\right)^n - \left(\frac{a}{r'}\right)^n\right\}\left\{\left(\frac{b}{r}\right)^n - \left(\frac{r}{b}\right)^n\right\}}{\left(\frac{b}{a}\right)^n - \left(\frac{a}{b}\right)^n} \quad (r' < r < b). \quad (15)$$

III. *Let* n *be a given positive number, and* $\rho_1, \rho_2, \ldots$ *the positive roots of the equation*

$$J_n(\rho a) = 0,$$

while κ^2 is not equal to ρ^2. Then we have

$$\sum_{\rho} \frac{J_n(\rho r)\, J_n(\rho r')}{(\rho^2-\kappa^2)\int_0^a r J_n^2(\rho r)\, dr} = e^{\frac{1}{2}n i\pi} \frac{J_n(\kappa r)}{J_n(\kappa a)} \{J_n(\kappa a)\, K_n(i\kappa r') - J_n(\kappa r')\, K_n(i\kappa a)\} \quad (0 < r < r')$$

$$= e^{\frac{1}{2}n i\pi} \frac{J_n(\kappa r')}{J_n(\kappa a)} \{J_n(\kappa a)\, K_n(i\kappa r) - J_n(\kappa r)\, K_n(i\kappa a)\} \quad (r' < r < b). \quad (16)$$

IV. *Let n be a given positive number, and $\rho_1, \rho_2, \ldots$ the positive roots of the equation*

$$U_n(\rho a) = J_n(\rho a)\, K_n(i\rho b) - J_n(\rho b)\, K_n(i\rho a) = 0 \quad (0 < a < b).$$

Then, κ^2 not being equal to ρ^2, we have

$$\sum_{\rho} \frac{U_n(\rho r)\, U_n(\rho r')}{(\rho^2-\kappa^2)\int_a^b r U_n^2(\rho r)\, dr}$$

$$= \frac{e^{\frac{1}{2}n i\pi} \{J_n(\kappa r)\, K_n(i\kappa a) - J_n(\kappa a)\, K_n(i\kappa r)\} \{J_n(\kappa r')\, K_n(i\kappa b) - J_n(\kappa b)\, K_n(i\kappa r')\}}{J_n(\kappa a)\, K_n(i\kappa b) - J_n(\kappa b)\, K_n(i\kappa a)} \quad (a < r < r')$$

$$= \frac{e^{\frac{1}{2}n i\pi} \{J_n(\kappa r')\, K_n(i\kappa a) - J_n(\kappa a)\, K_n(i\kappa r')\} \{J_n(\kappa r)\, K_n(i\kappa b) - J_n(\kappa b)\, K_n(i\kappa r)\}}{J_n(\kappa a)\, K_n(i\kappa b) - J_n(\kappa b)\, K_n(i\kappa a)} \quad (r' < r < b). \quad (17)$$

The first of these theorems was proved by Kneser (*Integralgleichungen*, § 27). The second follows in the same way. The third was proved by Kneser.* The fourth follows in the same way.

8. Using 7, I and II, the Green's function for the potential problems of § 4 reduce to:—

$$\text{I}: \frac{2}{a\sqrt{(rr')}} \sum_n \sum_m \frac{1}{2n+1} \frac{\left(\frac{r}{a}\right)^{n+\frac{1}{2}} \left\{\left(\frac{a}{r'}\right)^{n+\frac{1}{2}} - \left(\frac{r'}{a}\right)^{n+\frac{1}{2}}\right\}}{\int_{\mu_0}^1 [P_n^{-m\pi/a}(\mu)]^2\, d\mu}$$

$$\times P_n^{-m\pi/a}(\mu)\, P_n^{-m\pi/a}(\mu') \sin\frac{m\pi}{a}\phi \sin\frac{m\pi}{a}\phi', \quad (18)$$

* *Math. Ann.*, Bd. 63, p. 510 (1907).

when $0 < r < r'$.

$$\text{II}: \frac{2}{a\sqrt{(rr')}} \sum_n \sum_m \frac{\frac{1}{2n+1} \left\{ \left(\frac{r}{a}\right)^{n+\frac{1}{2}} - \left(\frac{a}{r}\right)^{n+\frac{1}{2}} \right\} \left\{ \left(\frac{b}{r'}\right)^{n+\frac{1}{2}} - \left(\frac{r'}{b}\right)^{n+\frac{1}{2}} \right\}}{\left\{ \left(\frac{b}{a}\right)^{n+\frac{1}{2}} - \left(\frac{a}{b}\right)^{n+\frac{1}{2}} \right\} \int_{\mu_0}^{1} [P_n^{-m\pi/\alpha}(\mu)]^2 \, d\mu}$$

$$\times P_n^{-m\pi/\alpha}(\mu) \, P_n^{-m\pi/\alpha}(\mu') \sin\frac{m\pi}{\alpha}\phi \sin\frac{m\pi}{\alpha}\phi', \quad (19)$$

where $a < r < r'$.

These results are completed by interchanging r and r', when $r' < r < a$ in I, and $r' < r < b$ in II.

9. Also using § 7, III and IV, the Green's functions for the equation $\nabla^2 u + \kappa^2 u = 0$ in the regions of § 6, I and II, reduce to:—

$$\text{I}: \frac{2}{a\sqrt{(rr')}} \sum_n \sum_m e^{\frac{1}{2}(n+\frac{1}{2})i\pi} \frac{J_{n+\frac{1}{2}}(\kappa r)}{J_{n+\frac{1}{2}}(\kappa a)} \{ J_{n+\frac{1}{2}}(\kappa a) \, K_{n+\frac{1}{2}}(i\kappa r') - J_{n+\frac{1}{2}}(\kappa r') \, K_{n+\frac{1}{2}}(i\kappa a) \}$$

$$\times \frac{P_n^{-m\pi/\alpha}(\mu) \, P_n^{-m\pi/\alpha}(\mu')}{\int_{\mu_0}^{1} [P_n^{-m\pi/\alpha}(\mu)]^2 \, d\mu} \sin\frac{m\pi}{\alpha}\phi \sin\frac{m\pi}{\alpha}\phi', \quad (20)$$

when $0 < r < r'$.

$$\text{II}: \frac{2}{a\sqrt{(rr')}} \sum_n \sum_m e^{\frac{1}{2}(n+\frac{1}{2})i\pi} \frac{\{ J_{n+\frac{1}{2}}(\kappa r) K_{n+\frac{1}{2}}(i\kappa a) - J_{n+\frac{1}{2}}(\kappa a) \, K_{n+\frac{1}{2}}(i\kappa r) \times \{ J_{n+\frac{1}{2}}(\kappa r') \, K_{n+\frac{1}{2}}(i\kappa b) - J_{n+\frac{1}{2}}(\kappa b) \, K_{n+\frac{1}{2}}(i\kappa r') \}}{J_{n+\frac{1}{2}}(\kappa a) \, K_{n+\frac{1}{2}}(i\kappa b) - J_{n+\frac{1}{2}}(\kappa b) \, K_{n+\frac{1}{2}}(i\kappa a)}$$

$$\times \frac{P_n^{-m\pi/\alpha}(\mu) \, P_n^{-m\pi/\alpha}(\mu')}{\int_{\mu_0}^{1} [P_n^{-m\pi/\alpha}(\mu)]^2 \, d\mu} \sin\frac{m\pi}{\alpha}\phi \sin\frac{m\pi}{\alpha}\phi', \quad (21)$$

when $a < r < r'$.

These results are completed by interchanging r and r', when $r' < r < a$ in I, and $r' < r < b$ in II.

Using the limiting values for the Bessel's functions $J_n(x)$ and $K_n(ix)$ as $x \to 0$, it will be found that the results I and II of this section reduce, when $\kappa \to 0$ to those of § 8, I and II.

This is an independent verification of the formulæ (20) and (21).

10. In his discussion of the problem of the diffraction of electric waves by a perfectly conducting right circular cone, Macdonald determined the constants in his series from the fact that these constants must be identical

with the constants in the solution of the corresponding potential problem, when both series are expressed in terms of the appropriate harmonic functions.

In his recent communication to these *Proceedings*, he puts in analytical form his argument for the general validity of this method.

In the solutions of §§ 8, 9 above, Macdonald's theorem holds.

One of the difficulties in the treatment of the equation $\nabla^2 u + \kappa^2 u = 0$ has been that we cannot proceed from the bounded region to the infinite region, as in the case of potential ; *e.g.* by letting the radius of the outer sphere tend to infinity in § 6, II. We have to make sure that there is no reflection at infinity. With Macdonald's method the solutions for the diffraction problems in infinite regions are obtained at once from those of the corresponding potential problems.

DIFFRACTION BY A WEDGE AND KINDRED PROBLEMS

By F. J. W. WHIPPLE.

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1. Dr. Bromwich's solution* of the problem of diffraction of waves by a wedge is based on the study of the special cases which can be solved by the method of images. To make the solution available for numerical work, the contour integrals used by Bromwich must be reduced to functions of real variables. This can be done by following the steps of his analysis in the reverse order.†

It is found that when waves are reflected from the inside or the outside of a wedge, the disturbance at any point is the sum of that due to the original waves, and the waves reflected according to the laws of geometrical optics, together with four trains of diffracted waves. The strength of each of these diffracted waves is given by a definite integral to which Fresnel's is a first approximation.

The reduction of the definite integrals to Fourier series presents some interesting features.

The paper concludes with a note on the diffraction of waves by an absorbing wedge. It is pointed out that Macdonald's solution of this problem leads to paradoxical results.

2. Bromwich's principal problem is to find a solution V of the equation

$$\left(\nabla^2 - c^2 \frac{d^2}{dt^2}\right) V = 0,$$

corresponding to a source placed at the point $r_0\theta_0 z_0$, between the two planes $\theta = 0$, $\theta = \pi/n$, subject to the condition that the solution is to vanish on both the boundary planes.

The disturbance near the source being $f(ct-R)/R$, the value of V is

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* *Proc. London Math. Soc.*, Ser. 2, Vol. 14, p. 450.

† [*Added May 22nd*, 1916.]—The process is equivalent to that adopted by Prof. Carslaw in solving the problem for the case of diffraction by a half-plane (*Proc. London Math. Soc.*, Ser. 1, Vol. xxx, pp. 141–144).

shewn to be given by $V = V_1 - V_2$, where

$$V_1 = \frac{1}{2\pi\iota}\int \frac{f(ct-R)}{R} G_1(\xi)\, d\xi,$$

$$V_2 = \frac{1}{2\pi\iota}\int \frac{f(ct-R)}{R} G_2(\xi)\, d\xi,$$

$$R^2 = r^2 + r_0^2 - 2rr_0 \cos\xi + (z-z_0)^2,$$

$$G_1(\xi) = \tfrac{1}{2}n \cot \tfrac{1}{2}n(\xi+\theta-\theta_0),$$

$$G_2(\xi) = \tfrac{1}{2}n \cot \tfrac{1}{2}n(\xi+\theta+\theta_0),$$

and the path of integration is that shewn in Fig. 1 (Bromwich's Fig. 3),

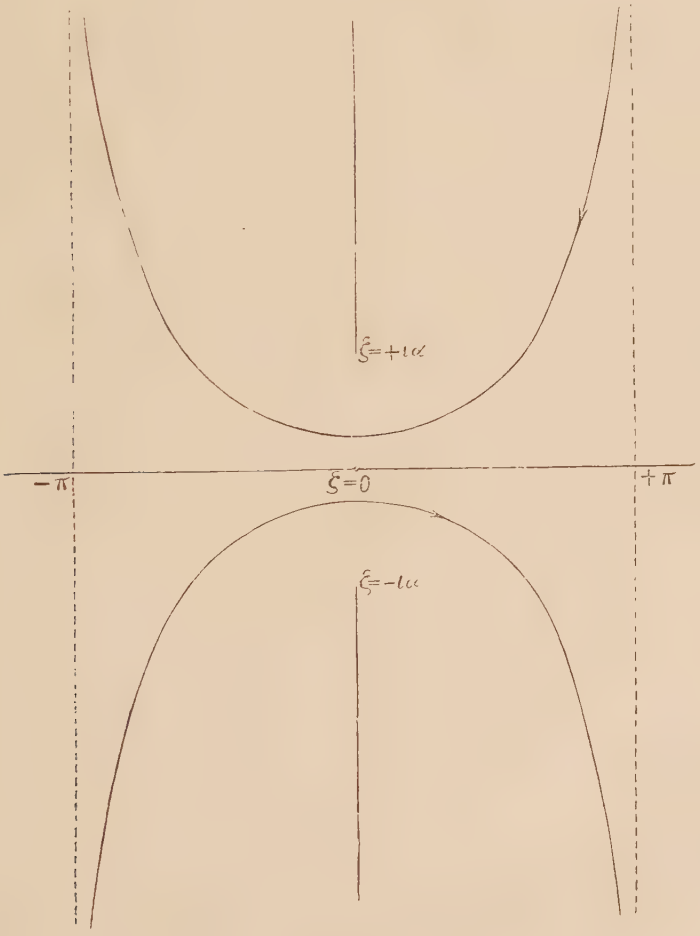

FIG. 1.

in which the branch points $\pm \iota\alpha$ are determined by

$$2rr_0 \cosh \alpha = r^2+r_0^2+(z-z_0)^2.$$

3. This path of integration may be deformed into two straight lines, and a loop as shewn in Fig. 2.

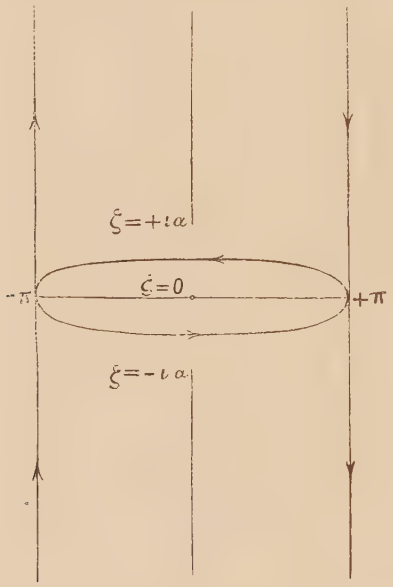

FIG. 2.

The loop contains as poles of the integral V_1 the roots of

$$\tan \frac{n}{2}(\zeta+\theta-\theta_0) = 0,$$

which lie between $\pm \pi$. The poles are given by

$$\zeta = \theta_0-\theta+2\kappa\pi/n,$$

where κ is integral, and corresponding to any one of these is a term in V_1, viz., $\frac{f(ct-R_{2\kappa})}{R_{2\kappa}}$, where

$$R_{2\kappa}^2 = r^2+r_0^2-2rr_0\cos(\theta_0-\theta+2\kappa\pi/n)+(z-z_0)^2.$$

Such a term represents a disturbance proceeding from one of the geometrical images of the source.

It is well known that these images lie on a circle. The set with azimuths $\theta_0+2\kappa\pi/n$ ($\kappa > 0$) are found by 2κ reflections, the last of which

is in the surface whose azimuth is π/n. The set with azimuths $\theta_0-2\kappa'\pi/n$ ($\kappa'>0$) are formed by $2\kappa'$ reflections, the last of which is in the surface whose azimuth is 0.

For an image to be visible from a certain point, the line joining the point and the image must intersect the surface at which the last reflection takes place. Accordingly an image of the former group is visible from a point whose azimuth is θ, provided that $\theta+\pi>\theta_0+2\kappa\pi/n$, and an image of the latter group is visible from the same point, provided that $\theta-\pi<\theta_0-2\kappa'\pi/n$. Thus the source and these two groups of visible images correspond exactly with the poles of the integral V_1.

In the same way it may be seen that the poles of the integral V_2 correspond with the images formed by an odd number of reflections in the two surfaces. The terms in V_2 associated with these poles are of the type $\frac{f(ct-R_{2\kappa-1})}{R_{2\kappa-1}}$, where

$$R_{2\kappa-1}^2=r^2+r_0^2-2rr_0\cos(-\theta_0-\theta+2\kappa\pi/n)+(z-z_0)^2.$$

4. To evaluate the part of V_1 obtained by integrating from $\pi+\iota\infty$ to $\pi-\iota\infty$ along the right-hand line in Fig. 2, we write

$$\zeta=\pi+\iota\eta,$$

$$R^2=r^2+r_0^2+2rr_0\cosh\eta+(z-z_0)^2=\rho^2+4rr_0\sinh^2\tfrac{1}{2}\eta,$$

where $$\rho^2=(r+r_0)^2+(z-z_0)^2,$$

so that ρ is the length of the shortest route from the point (r_0, z_0, θ_0) to the edge of the wedge, and thence to the point (r, θ, z).

$G_1(\zeta)$ takes the form

$$G_1(\zeta)=\tfrac{1}{2}n\cot\tfrac{1}{2}n(\pi+\theta-\theta_0+\iota\eta).$$

Now R is an even function of η, and therefore the two parts of the integral corresponding to $+\eta$ and $-\eta$ can be combined by adding the two values of G. The integral is then reduced to

$$-\frac{\frac{1}{2}n\sin n(\pi+\theta-\theta_0)}{2\pi}\int_0^\infty\frac{f(ct-R)}{R}\,\frac{d\eta}{\sin^2\frac{1}{2}n(\pi+\theta-\theta_0)+\sinh^2\frac{1}{2}n\eta},$$

where $$R^2=\rho^2+4rr_0\sinh^2\tfrac{1}{2}\eta.$$

In the same way the integral along the left-hand line in Fig. 2 is

$$-\frac{\frac{1}{2}n\sin n(\pi-\theta+\theta_0)}{2\pi}\int_0^\infty\frac{f(ct-R)}{R}\,\frac{d\eta}{\sin^2\frac{1}{2}n(\pi-\theta+\theta_0)+\sinh^2\frac{1}{2}n\eta}.$$

Similar analysis can be applied to the contour integral V_2, so that, finally,

$V =$ the sum of effects of visible images

$$-F(\pi+\theta-\theta_0)-F(\pi-\theta+\theta_0)+F(\pi-\theta-\theta_0)+F(\pi+\theta+\theta_0),$$

where the function F is defined by

$$F(\phi) = \frac{\frac{1}{2}n \sin n\phi}{2\pi} \int_0^\infty \frac{f(ct-R)}{R} \frac{d\eta}{\sin^2 \frac{1}{2}n\phi + \sinh^2 \frac{1}{2}n\eta},$$

and

$$R^2 \equiv \rho^2 + 4rr_0 \sinh^2 \tfrac{1}{2}\eta.$$

5. This is the solution of the problem. Before verifying that the conditions are satisfied, we notice that the form of $F(\phi)$ indicates that the disturbance at r, θ, z is built up of elementary waves which have taken times greater than ρ/c to reach that point. This view of the subject may be emphasized by taking T, the time at which the elementary wave leaves the source, as independent variable in the integral in $F(\phi)$. In fact,

$$F(\phi) = \frac{\frac{1}{2}n \sin n\phi}{2\pi} \int_{-\infty}^{t-\rho/c} \frac{f(cT)\, c\, dT}{rr_0 \sinh \eta\, (\sin^2 \frac{1}{2}n\phi + \sinh^2 \frac{1}{2}n\eta)},$$

where η is defined by

$$c^2 (t-T)^2 = \rho^2 + 4rr_0 \sinh^2 \tfrac{1}{2}\eta.$$

The integrand may be simplified when $t-T$ is large, so that the contribution to $F(\phi)$ from the waves which have been travelling a considerable time is

$$\frac{8(rr_0)^n}{c^{2n+1}} \frac{\frac{1}{2}n \sin n\phi}{2\pi} \int_{-\infty} \frac{f(cT)\, dT}{(t-T)^{2n+2}}.$$

It should be noticed, moreover, that the proposed solution is more general than Bromwich's, in that $f(ct-R)$ need not be defined for complex values of the variable. It will appear, however, that f and its first two differential coefficients must be continuous in order that the differential equation of the wave motion may be satisfied.

6. To verify that when $(r,\ \phi,\ z)$ are cylindrical coordinates, $F(\phi)$ satisfies the equation

$$\left[\nabla^2 - \frac{1}{c^2} \frac{d^2}{dt^2}\right] F = 0,$$

it is necessary to show that

$$\left[\left\{\frac{d^2}{dz^2}+\frac{1}{r}\frac{d}{dr}r\frac{d}{dr}-\frac{1}{c^2}\frac{d^2}{dt^2}\right\}-\frac{1}{r^2}\frac{d^2}{d\eta^2}\right]\frac{f(ct-R)}{R}=0,$$

and that

$$\left[\frac{d^2}{d\phi^2}+\frac{d^2}{d\eta^2}\right]\frac{\sin n\phi}{\sin^2\frac{1}{2}n\phi+\sinh^2\frac{1}{2}n\eta}=0.$$

This can be done without difficulty.

Then it follows that, if P be written for $\frac{f(ct-R)}{R}$ and Q for

$$\frac{\frac{1}{2}n\sin n\phi}{2\pi(\sin^2\frac{1}{2}n\phi+\sinh^2\frac{1}{2}n\eta)},$$

$$\begin{aligned}\left[\nabla^2-\frac{1}{c^2}\frac{d^2}{dt^2}\right]F &\equiv \left[\frac{d^2}{dz^2}+\frac{1}{r}\frac{d}{dr}r\frac{d}{dr}+\frac{1}{r^2}\frac{d^2}{d\phi^2}-\frac{1}{c^2}\frac{d^2}{dt^2}\right]F\\ &=\frac{1}{r^2}\int_0^\infty\left[Q\frac{d^2P}{d\eta^2}-P\frac{d^2Q}{d\eta^2}\right]d\eta\\ &=\frac{1}{r^2}\left[Q\frac{dP}{d\eta}-P\frac{dQ}{d\eta}\right]_0^\infty\\ &=0,\end{aligned}$$

i.e. F satisfies the wave equation as long as it is continuous.

The discontinuities of F occur where $\sin\frac{1}{2}n\phi=0$.

For simplicity of exposition we consider the discontinuity at $\phi=0$.

When ϕ passes through the value 0, the factor $\sin n\phi$ changes sign, the expression under the integral has an infinity, but it does not change sign.

For small positive values of ϕ the integral can be shewn to differ from

$$\frac{f(ct-\rho)}{\rho}\int_0^\infty\frac{d\eta}{\sin^2\frac{1}{2}n\phi+\frac{1}{4}n^2\eta^2}$$

by a quantity which remains finite as $\phi\to 0$. Evaluating the last integral we find that

$$F(+0)=\ \ \frac{1}{2}f(ct-\rho)/\rho.$$

Similarly

$$F(-0)=-\frac{1}{2}f(ct-\rho)/\rho.$$

Considerations of symmetry show that $\frac{\partial}{\partial\phi}F(\phi)$ is continuous through $\phi=0$.

The period of $F(\phi)$ is $2\pi/n$.

7. Returning to the wedge problem, we have to verify that the solution

$$V = [\Sigma f(ct-R_{2\kappa})/R_{2\kappa} - \Sigma f(ct-R_{2\kappa-1})/R_{2\kappa-1}]_{\text{visible}} \\ -F(\pi+\theta-\theta_0)-F(\pi-\theta+\theta_0)+F(\pi-\theta-\theta_0)+F(\pi+\theta+\theta_0),$$

satisfies the wave equation at all points where it is determinate, and that it satisfies the boundary conditions. We have also to find the form of V on the lines where the above expression is indeterminate.

(1) As each term of V is a solution of the wave equation, the sum must be a solution.

(2) At either of the boundaries $\theta = 0$, $\theta = \pi/n$,

$$F(\pi+\theta-\theta_0) = F(\pi-\theta-\theta_0),$$

and also

$$F(\pi+\theta+\theta_0) = F(\pi-\theta+\theta_0).$$

Moreover the images visible from a point very close to a boundary occur in pairs in such a way that the sum in the square bracket in the expression for V tends to zero at the boundary.

Accordingly the boundary condition $V = 0$ is satisfied.

(3) The above expression for V is indeterminate at points which lie on the limiting lines of the shadows thrown by the various images.

Consider, for example, the image (2κ) which lies on the radius $\theta_0+2\kappa\pi/n$. This image is invisible at points for which $\theta+\pi < \theta_0+2\kappa\pi/n$, and visible at points for which $\theta+\pi > \theta_0+2\kappa\pi/n$. Now as θ approaches $(\theta_0+2\kappa\pi/n-\pi)$ from below $F(\pi+\theta-\theta_0)$ tends to $F(-0)$, which is $-\frac{1}{2}f(ct-\rho)/\rho$. As θ approaches the same value from above $F(\pi+\theta-\theta_0)$ tends to $F(+0)$, which is $+\frac{1}{2}f(ct-\rho)/\rho$ and $f(ct-R_{2\kappa})/R_{2\kappa}-F(\pi+\theta-\theta_0)$ tends to $f(ct-\rho)/\rho-\frac{1}{2}f(ct-\rho)/\rho$, which is equal to $\frac{1}{2}f(ct-\rho)/\rho$.

The other terms in V present no difficulty, and we are therefore led to take

$$V = [\Sigma f(ct-R_{2\kappa-2})/R_{2\kappa-2} - \Sigma f(ct-R_{2\kappa-1})/R_{2\kappa-1}]^{\text{quite}}_{\text{visible}} \\ +\tfrac{1}{2}f(ct-R_{2\kappa})/R_{2\kappa}-F(\pi-\theta+\theta_0)+F(\pi-\theta-\theta_0)+F(\pi+\theta+\theta_0),$$

as the value of V on the shadow limit.

By so doing we secure the continuity of V and dV/dr. Moreover, $dV/rd\theta$ is also continuous, since, as has been noted already, $\partial F/\partial\phi$ is continuous across $\phi = 0$ and $\partial/\partial\theta[f(ct-R_{2\kappa})/R_{2\kappa}]$ vanishes on the limiting plane.

Similar considerations apply to the limits of the regions of visibility

of the other images. It is convenient to tabulate the discontinuities of the four functions F,

$F(\pi+\theta-\theta_0)$	Images (2κ) formed by 2κ reflections, the last in π/n. Also the source if $\theta_0 > \pi$.
$F(\pi-\theta+\theta_0)$	Images $(-2\kappa')$ formed by $2\kappa'$ reflections, the last in 0. Also the source if $n < 1$, $\theta_0 < \pi/n-\pi$.
$F(\pi-\theta-\theta_0)$	Images $(-2\kappa-1)$ formed by $2\kappa+1$ reflections, the last in 0.
$F(\pi+\theta+\theta_0)$	Images $(2\kappa-1)$ formed by $2\kappa-1$ reflections, the last in π/n.

The corresponding limiting forms of V can be written down at once. In each case the sign in front of the $\frac{1}{2}$ is opposite to the sign in front of the suppressed F.

8. With the trivial exceptions which occur when n is an integer, and when one of the images is exactly on one of the bounding planes of the wedge, there are two and only two discontinuities in every example. They may occur in any two of the four functions F.

For instance, we may take the case of the right-angled wedge cited by Dr. Bromwich (*l.c.*, p. 454).

The three figures correspond to the three quadrants in which the source may lie.

In Fig. 3, $n = \frac{2}{3}$, $\theta_0 < \pi/2$.

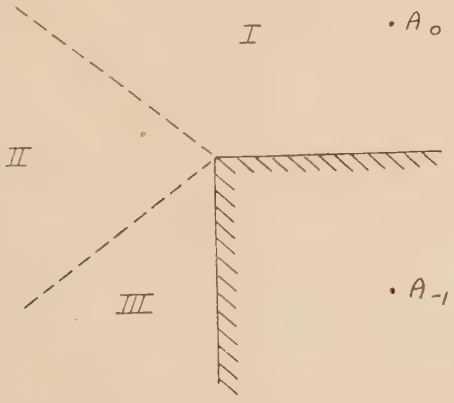

FIG 3

The source A_0 and the image A_{-1} are visible in region I, A_0 alone is visible in region II, and neither in region III.

On the boundary between I and II, there is a discontinuity in $F(\pi-\theta-\theta_0)$; on the boundary II, III in $F(\pi-\theta+\theta_0)$.

In Fig. 4, $n=\frac{2}{3}$, $\frac{\pi}{2}<\theta_0<\pi$.

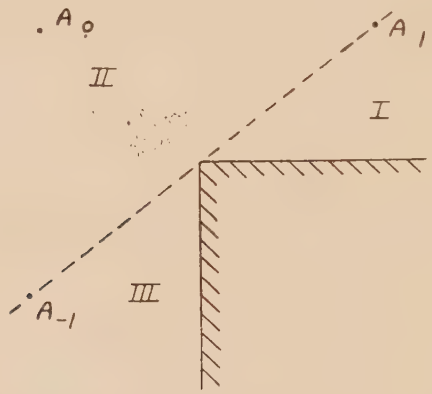

FIG 4

The source A_0 is visible everywhere, but A_1 is only visible in region III, A_{-1} in region I.

On the boundary I, II there is a discontinuity in $F(\pi-\theta-\theta_0)$, on the boundary II, III in $F(\pi+\theta+\theta_0)$.

In Fig. 5, $n=\frac{2}{3}$, $\theta>\pi$.

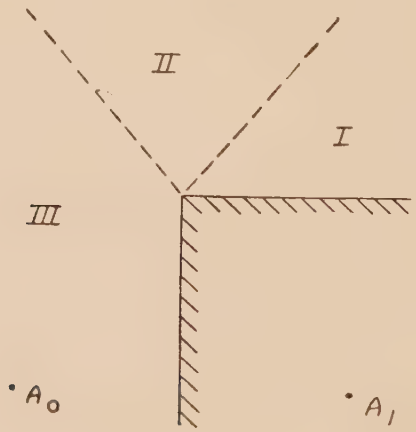

FIG 5

There are discontinuities in $F(\pi+\theta-\theta_0)$ and $F(\pi+\theta+\theta_0)$.

9. It will now be shewn that when the source is emitting a pure tone the function $F(\phi)$ may, for small wave-lengths, be expressed in terms of Fresnel's integrals.

Let $$f(ct-R) = \exp 2\pi\iota(ct-R)/\lambda,$$

Then $$F(\phi) = \frac{\frac{1}{2}n \sin n\phi}{2\pi} \int_0^\infty \frac{\exp 2\pi\iota(ct-R)/\lambda}{R} \frac{d\eta}{\sin^2 \frac{1}{2}n\phi + \sinh^2 \frac{1}{2}n\eta},$$

where R stands for $[\rho^2 + 4rr_0 \sinh^2 \frac{1}{2}\eta]^{\frac{1}{2}}$.

Now, if the wave-length λ is very small compared with the other lengths involved, the contributions to the integral of successive half-periods of the exponential cancel very nearly, so that the most important part of $F(\phi)$ is contributed by small values of η, and we may write*

$$F(\phi) \doteqdot \frac{\frac{1}{2}n \sin n\phi}{2\pi} \frac{\exp 2\pi\iota(ct-\rho)/\lambda}{\rho} \int_0^\infty \exp\left(-\frac{2\pi\iota}{\lambda} \frac{2rr_0}{\rho} \frac{\eta^2}{4}\right) \frac{d\eta}{\sin^2 \frac{1}{2}n\phi + \frac{1}{4}n^2\eta^2}$$

$$\doteqdot \frac{\sin n\phi}{n\pi} \frac{\exp 2\pi\iota(ct-\rho)/\lambda}{\rho} \int_0^\infty \exp(-\iota q^2\eta^2) \frac{d\eta}{\psi^2+\eta^2},$$

where $$\psi = \frac{2}{n} \sin \tfrac{1}{2}n\phi,$$

$$q^2 = rr_0\pi/\lambda\rho.$$

Now $$\int_0^\infty \exp(-\iota q^2\eta^2) \frac{d\eta}{\psi^2+\eta^2}$$

$$= \exp \iota q^2\psi^2 \int_0^\infty \exp[-\iota q^2(\psi^2+\eta^2)] \frac{d\eta}{\psi^2+\eta^2}$$

$$= \exp \iota q^2\psi^2 \int_0^\infty \int_q^\infty \exp[-\iota x^2(\psi^2+\eta^2)]\,(2\iota x)\,dx\,d\eta$$

$$\doteqdot \exp\left(\iota q^2\psi^2 + \iota \frac{\pi}{4}\right)\sqrt{\pi} \int_q^\infty \exp[-\iota x^2\psi^2]\,dx.$$

Thus

$$F(\phi) \doteqdot \frac{1}{\sqrt{\pi}} \frac{\sin n\phi}{n\psi} \frac{\exp\{2\pi\iota(ct-\rho)/\lambda + \iota q^2\psi^2 + \iota\pi/4\}}{\rho} \int_{q\psi}^\infty \exp(-\iota u^2)\,du,$$

and the integral is of Fresnel's type.

For small values of ϕ,

$$F(\phi) \doteqdot \frac{1}{\sqrt{\pi}} \frac{\exp\{2\pi\iota(ct-\rho)\lambda + \iota q^2\phi^2 + \iota\pi/4\}}{\rho} \int_{q\phi}^\infty \exp(-\iota u^2)\,du,$$

an expression in which n does not appear.

* We use the symbol $\doteqdot$ to express approximate equality.

10. This approximation to Fresnel's integrals is only valid when the wave-length is small compared with the distances r, r_0.

It would be interesting to compute the values of $F(\phi)$ in other cases; for example, the hydrodynamical problem of waves beating against a wedge-shaped breakwater might be worked out in detail.

11. The expression $F(\phi)$ can be split up into a Fourier series with ϕ as variable by making use of the expansion

$$\frac{\sin n\phi}{2\cosh nu - 2\cos n\phi} = \sum_{m=1}^{\infty} e^{-nmu} \sin nm\phi.$$

Thus $$F(\phi) = \Sigma Z_m \sin nm\phi,$$

where $$Z_m = \frac{n}{\pi} \int_0^{\infty} \frac{f[ct-(\rho^2+4rr_0\sinh^2\frac{1}{2}u)^{\frac{1}{2}}]}{(\rho^2+4rr_0\sinh^2\frac{1}{2}u)^{\frac{1}{2}}} e^{-nmu}\, du.$$

It might be anticipated that $Z_m \sin nm\phi$ would be a normal function satisfying the wave equation. This is not the case however.

If Ue^{-nmu} be written for the subject of integration, it can be shewn that

$$\left[\nabla^2 - \frac{1}{c^2}\frac{d^2}{dt^2}\right] U = \frac{1}{r^2}\frac{d^2}{du^2} U.$$

Hence

$$\left[\nabla^2 - \frac{1}{c^2}\frac{d^2}{dt^2}\right] Z_m \sin nm\phi = \frac{n\sin nm\phi}{\pi} \frac{1}{r^2} \int_0^{\infty} \left\{\frac{d^2U}{du^2} - n^2m^2U\right\} e^{-nmu}\, du$$

$$= \frac{n\sin nm\phi}{\pi} \frac{1}{r^2} \left[\left(\frac{dU}{du} + nmU\right) e^{-nmu}\right]_0^{\infty}$$

$$= -\frac{n^2m}{\pi} \frac{1}{r^2} \frac{f(ct-\rho)}{\rho} \sin nm\phi.$$

The paradox that $F(\phi)$ satisfies the wave equation whilst its constituent terms do not, is to be associated with the fact that the "sum" of the divergent series $\frac{n^2}{\pi r^2} \frac{f(ct-\rho)}{\rho} \Sigma m \sin mn\phi$ is zero.

It should not be forgotten that $F(\phi)$ has a discontinuity on the plane $\phi = 0$, so that it is not a universal solution of the wave equation.*

The complete solution of the diffraction problem can, however, be re-

* [*Added May 22nd*, 1916.]—Cf. Stokes, "On the Critical Values of the Sums of Periodic Series," *Collected Papers*, Vol. 1, pp. 256–8. I owe this reference to Dr. Bromwich.

duced to normal functions of this type. The process virtually consists in retracing the steps in the earlier part of this paper.

We have

$$F(\pi+\theta-\theta_0)+F(\pi-\theta+\theta_0)-F(\pi+\theta+\theta_0)-F(\pi-\theta-\theta_0)$$
$$= 4\Sigma \sin nm\pi \sin nm\theta \sin nm\theta_0 Z_m.$$

The contribution to V due to the direct rays from the source and the images may be expressed as a Fourier series of sines. To determine the coefficient of $\sin nm\theta$ in this series we have to evaluate

$$\frac{n}{\pi}\int_0^{\pi/n}\left[\frac{f(ct-R)}{R}+\Sigma(-)^q\frac{f(ct-R_q)}{R_q}\right]\sin nm\theta\, d\theta.$$

The part of this integrand referring to any image is only to be included over the arc where that image is visible.

The last expression can be transformed into

$$\frac{n}{\pi}\int_{\theta_0-\pi}^{\theta_0+\pi}\frac{f(ct-R)}{R}\sin nm\theta\, d\theta,$$

where $$R^2 = r^2+r_0^2-2rr_0\cos(\theta-\theta_0)+(z-z_0)^2,$$

which is equivalent to

$$\frac{2n}{\pi}\sin nm\theta_0\int_{-\pi}^{\pi}\frac{f(ct-R)}{R}\cos nm\psi\, d\psi,$$

where $$R^2 = r^2+r_0^2-2rr_0\cos\psi+(z-z_0)^2.$$

Hence the disturbance due to the source and images is

$$4\Sigma \sin nm\theta \sin nm\theta_0\, Y_m,$$

where $$Y_m = \frac{n}{\pi}\int_0^{\pi}\frac{f(ct-R)}{R}\cos nm\psi\, d\psi.$$

The normal function is evidently

$$\sin nm\theta\,[Y_m-\sin nm\pi Z_m].$$

12. *Special Case* 1: Plane waves with the direction of propagation at right angles to the edge of the wedge.

In the case of simple harmonic motion, the initial disturbances may be represented by $\exp[\iota\kappa ct+\iota\kappa r\cos(\theta-\theta_0)]$, and the diffracted trains

of waves by functions of the type

$$F(\phi) \equiv \frac{n \sin n\phi}{2\pi} \exp(\iota\kappa ct) \int_0^\infty e^{-\iota\kappa r \cosh u} \frac{du}{\cosh nu - \cos n\phi}.$$

Thus
$$Z_m = \frac{n}{\pi} e^{\iota\kappa ct} \int_0^\infty e^{-\iota\kappa r \cosh u} e^{-nmu}\, du,$$

and
$$Y_m = \frac{n}{\pi} e^{\iota\kappa ct} \int_0^\pi e^{\iota\kappa r \cos\psi} \cos nm\psi\, d\psi.$$

It follows that

$$Y_m - \sin nm\pi Z_m = n e^{\iota\kappa ct + \iota nm\pi/2} J_{nm}(\kappa r),$$

so that the solution is reduced to Bessel functions.*

It appears likely that for the purposes of numerical calculation the functions Z_m would be more useful than the J_{nm}'s.

13. *Special Case* 2.

The problem of the diffraction of a plane wave of arbitrary type has been solved by Lamb† for the case of a straight edge ($n = \frac{1}{2}$).

To obtain comparable formulæ we represent the initial disturbance by $\psi[ct + r\cos(\theta - \theta_0)]$. One pair of terms representing diffracted waves will be

$$\begin{aligned}
&F(\pi - \theta + \theta_0) + F(\pi + \theta - \theta_0) \\
&\equiv F(\phi) + F(2\pi - \phi) \\
&= \frac{\frac{1}{2}\sin\frac{1}{2}\phi}{2\pi} \int_0^\infty \psi(ct - r\cosh u) \left\{ \frac{1}{\cosh\frac{u}{2} - \cos\frac{\phi}{2}} + \frac{1}{\cosh\frac{u}{2} + \cos\frac{\phi}{2}} \right\} du \\
&= \frac{\sin\frac{1}{2}\phi}{2\pi} \int_0^\infty \psi(ct - r\cosh u) \frac{\cosh\frac{u}{2}}{\cosh^2\frac{u}{2} - \cos^2\frac{\phi}{2}} du \\
&= \sqrt{(2r)}\, \frac{\sin\frac{1}{2}\phi}{2\pi} \int_{-\infty}^{t - r/c} \frac{\psi(cT)}{ct - cT - r\cos\phi} \frac{c\, dT}{(ct - cT - r)^{\frac{1}{2}}}.
\end{aligned}$$

This formula shews that no diffracted disturbance reaches any point until there has been time for the wave to be reflected to it from the edge of the wedge. Lamb's solution, which is given as a triple integral, can be reduced

* Cf. Jackson, *Proc. London Math. Soc.*, Ser. 2, Vol. 1, 1903, p. 396.

† *Proc. London Math. Soc.*, Ser. 2, Vol. 8, p. 422.

to this form. In his paper the diffraction of a solitary wave is discussed in detail.

14. *Special Case* 3.

The case of an electrostatic point-charge in the neighbourhood of a conducting wedge is represented by taking unity instead of $f(ct-R)$ in the formula of § 2.

It follows that

$$F(\phi) = \frac{n \sin n\phi}{2\pi} \int_0^\infty \frac{1}{(\rho^2+4rr_0 \sinh^2 \frac{1}{2}u)^{\frac{1}{2}}} \frac{du}{\cosh nu - \cos n\phi},$$

$$Z_m = \frac{n}{\pi} \int_0^\infty \frac{1}{(\rho^2+4rr_0 \sinh^2 \frac{1}{2}u)^{\frac{1}{2}}} e^{-nmu}\, du,$$

$$Y_m = \frac{n}{\pi} \int_0^\pi \frac{1}{(\rho^2-4rr_0 \cos^2 \frac{1}{2}\psi)^{\frac{1}{2}}} \cos nm\psi\, d\psi.$$

We have to identify $Y_m - \sin nm\pi Z_m$.

It will be found that this expression reduces to a toroidal function, and it is therefore convenient to introduce the coordinates of suitable type at this stage.

Consider a circle whose centre is on the edge of the wedge, whose plane is perpendicular to the edge and which passes through the source (or charged point) P_0. Let the plane through P and the edge cut this circle at A and B. The toroidal coordinates of P are

$$\xi = \log PB/PA, \qquad \eta = \angle BPA,$$

and θ the azimuth. We have at once

$$\coth \xi = \frac{PB^2+PA^2}{PB^2-PA^2} = \frac{\rho^2-2rr_0}{2rr_0},$$

Hence

$$Z_m = \frac{1}{\sqrt{(rr_0)}} \frac{n}{\pi} \int_0^\infty \frac{1}{(2 \coth \xi + 2 \cosh u)^{\frac{1}{2}}} e^{-nmu}\, du,$$

$$Y_m = \frac{1}{\sqrt{(rr_0)}} \frac{n}{\pi} \int_0^\pi \frac{1}{(2 \coth \xi - 2 \cos \psi)^{\frac{1}{2}}} \cos nm\psi\, d\psi.$$

Y_m is known to be a multiple of a toroidal harmonic $P_{-\frac{1}{2}}^{-nm}(\cosh \xi)$ if nm is integral.* The more general formula which must be used when nm is fractional requires special investigation.

* Hobson, *Phil. Trans.*, 187, A, 1896, p. 524.

15. In the following section m is used instead of mn for the rank of a harmonic function, while n occurs as the order of the function and is not to be confused with the n of the wedge problem.

Hobson's generalized definition of the Legendre function is [*l.c.*, p. 451 (4)]

$$P_n^{-m}(\mu) = \frac{e^{-n\pi\iota}}{4\pi \sin n\pi} \frac{1}{2^n} \frac{\Pi(n-m)}{\Pi(n)} (\mu^2-1)^{-\frac{1}{2}m} \int^{\mu+,\,1+,\,\mu-,\,1-} \frac{(t^2-1)^n}{(t-\mu)^{n-m+1}}\, dt.$$

We consider only the case when $\mu > 1$, $m > n$, and write $\mu = \cosh \xi$.

The contour of the integral consists of two loops round the point μ, and two round the point 1. This contour can be deformed into that shewn in Fig. 6, and ultimately into the circle with radius $\sinh \xi$ and centre $\cosh \xi$, the real axis between $\cosh \xi$ and $\cosh \xi + \sinh \xi$ and the infinitesimal circle round $\cosh \xi$.

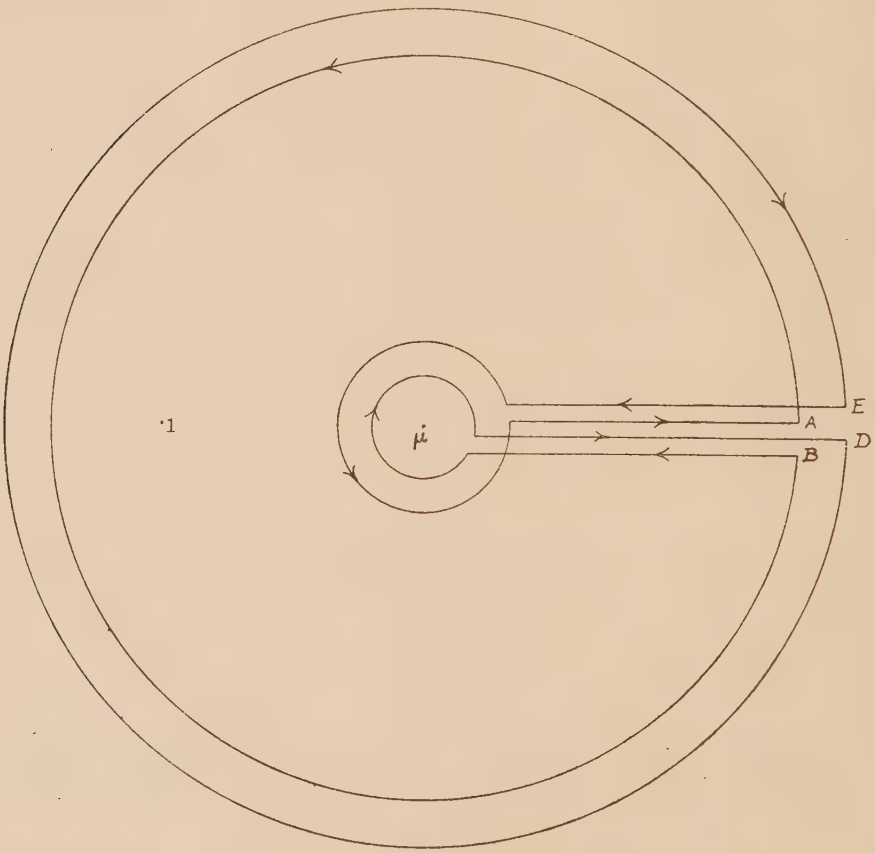

FIG. 6.

The phases of the powers are such that they vanish initially at A on the real axis. For points on the circle of radius $\sinh \xi$ we may write

$$t = \cosh \xi - e^{\iota\psi} \sinh \xi,$$

so that $$t^2 - 1 = -2e^{\iota\psi} \sinh \xi (\cosh \xi - \sinh \xi \cos \psi),$$

and ψ increases from $-\pi$ at A to $+\pi$ at B.

For points on the axis write

$$t = \cosh \xi + e^{-u} \sinh \xi,$$

$$t^2 - 1 = 2e^{-u} \sinh \xi (\cosh \xi + \sinh \xi \cosh u).$$

Further, let

$$X = 2^n \sinh^m \xi \int_{-\pi}^{\pi} e^{m\iota\psi} (\cosh \xi - \sinh \xi \cos \psi)^n \, d\psi,$$

$$Y = 2^n \sinh^m \xi \int_0^{\infty} e^{-mu} (\cosh \xi + \sinh \xi \cosh u)^n \, du,$$

and notice that the contour integral round the infinitesimal circle $t = \mu$ vanishes, since $m > n$. The phase of the subject of integration in the definition of $P_n^{-m}(\mu)$ is 0 at A, $2(m-1)\pi$ at B, $2n\pi$ at D, and $2(n-m+1)\pi$ at E. Thus the integral is

$$e^{m\iota\pi}\iota X - e^{2(m-1)\iota\pi} Y + e^{2n\iota\pi} Y - e^{(2n-m)\iota\pi}\iota X - e^{2(n-m+1)\iota\pi} Y + Y$$

$$= 4 \sin (m-n)\pi . \sin m\pi . e^{n\iota\pi} Y - 2 \sin (m-n)\pi . e^{n\iota\pi} X.$$

Hence

$$P_n^{-m}(\cosh \xi) = -\frac{1}{2^n \sinh^m \xi} \frac{\Pi(n-m)}{\Pi(n)} \frac{\sin (m-n)\pi}{\sin n\pi} \left[\frac{1}{2\pi} X - \frac{\sin m\pi}{\pi} Y \right],$$

or $$P_n^{-m}(\cosh \xi) = \frac{1}{\pi} \frac{\Pi(-1-n)}{\Pi(m-1-n)} \left\{ \int_0^{\pi} (\cosh \xi - \sinh \xi \cos \psi)^n \cos m\psi \, d\psi \right.$$

$$\left. - \sin m\pi \int_0^{\infty} (\cosh \xi + \sinh \xi \cosh u)^n \, e^{-mu} \, du \right\}.$$

This formula holds good as long as $m > n$. In particular, if $n = -\frac{1}{2}$ and m is positive,

$$P_{-\frac{1}{2}}^{-m}(\cosh \xi) = \frac{1}{\pi} \frac{\Pi(-\frac{1}{2})}{\Pi(m-\frac{1}{2})} \left\{ \int_0^{\pi} (\cosh \xi - \sinh \xi \cos \psi)^{-\frac{1}{2}} \cos m\psi \, d\psi \right.$$

$$\left. - \sin m\pi \int_0^{\infty} (\cosh \xi + \sinh \xi \cosh u)^{-\frac{1}{2}} \, e^{-mu} \, du \right\}.$$

16. Returning to the formulæ of § 14, we find that

$$Y_m - \sin nm\pi Z_m = \frac{n}{(2rr_0/\sinh\xi)^{\frac{1}{2}}} \frac{\Pi(mn-\frac{1}{2})}{\Pi(-\frac{1}{2})} P_{-\frac{1}{2}}^{-nm}(\cosh\xi).$$

Thus the potential due to an electric charge E at a point between two conducting planes is expressed in the form

$$\frac{E.n}{(2rr_0/\sinh\xi)^{\frac{1}{2}}} \sum_1^\infty 4 \sin m\theta \sin m\theta_0 \frac{\Pi(mn-\frac{1}{2})}{\Pi(-\frac{1}{2})} P_{-\frac{1}{2}}^{-nm}(\cosh\xi).$$

It may be noticed that Macdonald's analysis of this problem provides formulæ for the expansion of toroidal functions in terms of spherical or cylindrical harmonics. Thus*

$$\frac{1}{(2\varpi\varpi_0/\sinh\xi)^{\frac{1}{2}}} P_{-\frac{1}{2}}^{-m}(\cosh\xi) = \frac{\Pi(-\frac{1}{2})}{\Pi(m-\frac{1}{2})} \sum_{n=m}^\infty \frac{\Pi(n+m)}{\Pi(n-m)} \frac{r^n}{r_0^{n+1}} P_n^{-m}(\mu) P_n^{-m}(\mu_0),$$

where n takes values such that $n-m$ is a positive integer.

In this formula

$$\varpi = r\sin\theta, \quad \varpi_0 = r_0\sin\theta_0, \quad \mu = \cos\theta, \quad \mu_0 = \cos\theta_0,$$

and

$$\coth\xi = \frac{r^2+r_0^2-2rr_0\mu\mu_0}{2rr_0\sin\theta\sin\theta_0}$$

determines ξ. And on the other hand†

$$\frac{1}{(2\varpi\varpi_0/\sinh\xi)^{\frac{1}{2}}} P_{-\frac{1}{2}}^{-m}(\cosh\xi) = \frac{\Pi(-\frac{1}{2})}{\Pi(m-\frac{1}{2})} \int_0^\infty e^{-\lambda z} J_m(\lambda\varpi) J_m(\lambda\varpi_0)\, d\lambda,$$

where

$$\coth\xi = \frac{\varpi^2+\varpi_0^2+z^2}{2\varpi\varpi_0}.$$

17. The problem of the diffraction of electric waves by an absorbing wedge has been discussed by Macdonald.‡ He takes the actual disturbance to be the arithmetic mean of that which would exist if the surfaces of the wedge were perfect conductors, and that which would exist if they were incapable of supporting magnetic force.

In the notation of § 2, the two solutions for the case of a wave from a

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* *Proc. London Math. Soc.*, Ser. 2, Vol. 14, p. 416.

† *Proc. London Math. Soc.*, Ser. 1, Vol. xxvi, p. 160.

‡ *Proc. London Math. Soc.*, Ser. 2, Vol. 12, p. 430.

source producing electric force parallel to the edge of the wedge are

$$V = V_1 - V_2 \quad \text{and} \quad V = V_1 + V_2,$$

so that the mean disturbance is given by $V = V_1$.

In the notation of §§ 4, 7, this solution is

$$V = [\Sigma f(ct - R_{2\kappa})/R_{2\kappa}]_{\text{visible}} - F(\pi + \theta - \theta_0) - F(\pi - \theta + \theta_0).$$

With a true wedge this formula is simplified, as the only possible value of κ is zero.

There seems to be no reason to confine it to this case. When applied to the problem of the source between two absorbing planes, it implies that although images formed by one reflection do not exist, those formed by two reflections are as bright as the source. This paradoxical result suggests that the hypothesis on which it is based may require revision, and therefore makes the validity of the solution for the true wedge open to question.

ASYMPTOTIC FORMULÆ FOR THE DISTRIBUTION OF INTEGERS OF VARIOUS TYPES*

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1.

Statement of the problem.

1.1. We denote by q a number of the form

$$2^{a_2}\,3^{a_3}\,5^{a_5}\ldots p^{a_p}, \tag{1.11}$$

where 2, 3, 5, ..., p are primes and

$$a_2 \geqslant a_3 \geqslant a_5 \ldots \geqslant a_p\,; \tag{1.111}$$

and by $Q(x)$ the number of such numbers which do not exceed x: and our problem is that of determining the order of $Q(x)$. We prove that

$$Q(x) = \exp\left[\{1+o(1)\}\,\frac{2\pi}{\sqrt{3}}\sqrt{\left(\frac{\log x}{\log\log x}\right)}\right], \tag{1.12}$$

that is to say that to every positive ϵ corresponds an $x_0 = x_0(\epsilon)$, such that

$$\left(\frac{2\pi}{\sqrt{3}}-\epsilon\right)\sqrt{\left(\frac{\log x}{\log\log x}\right)} < \log Q(x) < \left(\frac{2\pi}{\sqrt{3}}+\epsilon\right)\sqrt{\left(\frac{\log x}{\log\log x}\right)}, \tag{1.121}$$

for $x > x_0$. These functions are of course of higher order than any power of $\log x$, but of lower order than any power of x.

The interest of the problem is threefold. In the first place the result itself, and the method by which it is obtained, are curious and interesting in themselves. Secondly, the method of proof is one which, as we show at the end of the paper, may be applied to a whole class of problems in the analytic theory of numbers: it enables us, for example, to find

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* This paper was originally communicated under the title "A problem in the Analytic Theory of Numbers".

asymptotic formulæ for the number of partitions of n into positive integers, or into different positive integers, or into primes. Finally, the class of numbers q includes as a sub-class the "highly composite" numbers recently studied by Mr. Ramanujan in an elaborate memoir in these *Proceedings*.* The problem of determining, with any precision, the number $H(x)$ of highly composite numbers not exceeding x appears to be one of extreme difficulty. Mr. Ramanujan has proved, by elementary methods, that the order of $H(x)$ is at any rate greater than that of $\log x$† : but it is still uncertain whether or no the order of $H(x)$ is greater than that of any power of $\log x$. In order to apply transcendental methods to this problem, it would be necessary to study the properties of the function

$$\mathfrak{H}(s) = \Sigma \frac{1}{h^s},$$

where h is a highly composite number, and we have not been able to make any progress in this direction. It is therefore very desirable to study the distribution of wider classes of numbers which include the highly composite numbers and possess some at any rate of their characteristic properties. The simplest and most natural such class is that of the numbers q ; and here progress is comparatively easy, since the function

$$\mathfrak{Q}(s) = \Sigma \frac{1}{q^s} \tag{1.13}$$

possesses a product expression analogous to Euler's product expression for $\zeta(s)$, viz.

$$Q(s) = \prod_1^\infty \left(\frac{1}{1-l_n^{-s}}\right), \tag{1.14}$$

where $l_n = 2.3.5 \ldots p_n$ is the product of the first n primes.

We have not been able to apply to this problem the methods, depending on the theory of functions of a complex variable, by which the Prime Number Theorem was proved. The function $Q(s)$ has the line $\sigma = 0$‡ as a line of essential singularities, and we are not able to obtain sufficiently accurate

* Ramanujan, "Highly Composite Numbers", *Proc. London Math. Soc.*, Ser. 2, Vol. 14, 1915, pp. 347–409.

† As great as that of $\dfrac{\log x \sqrt{(\log\log x)}}{(\log\log\log x)^2}$: see p. 385 of his memoir.

‡ We write as usual $s = \sigma + it$.

information concerning the nature of these singularities. But it is easy enough to determine the behaviour of $Q(s)$ as a function of the *real* variable s; and it proves sufficient for our purpose to determine an asymptotic formula for $Q(s)$ when $s \to 0$, and then to apply a "Tauberian" theorem similar to those proved by Messrs. Hardy and Littlewood in a series of papers published in these *Proceedings* and elsewhere.*

This "Tauberian" theorem is in itself of considerable interest as being (so far as we are aware) the first such theorem which deals with functions or sequences tending to infinity more rapidly than any power of the variable.

2.

Elementary results.

2.1. Let us consider, before proceeding further, what information concerning the order of $Q(x)$ can be obtained by purely elementary methods.

Let

$$l_n = 2 \cdot 3 \cdot 5 \ldots p_n = e^{\vartheta(p_n)}, \tag{2.11}$$

where $\vartheta(x)$ is Tschebyschef's function

$$\vartheta(x) = \sum_{p \leqslant x} \log p.$$

The class of numbers q is plainly identical with the class of numbers of the form

$$l_1^{b_1} l_2^{b_2} \ldots l_n^{b_n}, \tag{2.12}$$

where $$b_1 \geqslant 0, \quad b_2 \geqslant 0, \quad \ldots, \quad b_n \geqslant 0.$$

Now every b can be expressed in one and only one way in the form

$$b_i = c_{i,m} 2^m + c_{i,m-1} 2^{m-1} + \ldots + c_{i,0}, \tag{2.13}$$

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* See, in particular, Hardy and Littlewood, "Tauberian theorems concerning power series and Dirichlet's series whose coefficients are positive", *Proc. London Math. Soc.*, Ser. 2, Vol. 13, pp. 174–191; and "Some theorems concerning Dirichlet's series", *Messenger of Mathematics*, Vol. 43, pp. 134–147.

where every c is equal to zero or to unity. We have therefore

$$q = \prod_{i=1}^{n} \left(l_i^{\sum_{j=0}^{m} c_{i,j} 2^j} \right) = \prod_{j=0}^{m} \prod_{i=1}^{n} l_i^{c_{i,j} 2^j} = \prod_{j=0}^{m} r_j^{2^j}, \tag{2.14}$$

say, where

$$r_j = l_1^{c_{1,j}} l_2^{c_{2,j}} \dots l_n^{c_{n,j}}. \tag{2.141}$$

Let r denote, generally, a number of the form

$$r = l_1^{c_1} l_2^{c_2} \dots l_n^{c_n}, \tag{2.15}$$

where every c is zero or unity : and $R(x)$ the number of such numbers which do not exceed x. If $q \leqslant x$, we have

$$r_0 \leqslant x, \quad r_1^2 \leqslant x, \quad r_2^4 \leqslant x, \quad \dots .$$

The number of possible values of r_0, in the formula (2.14), cannot therefore exceed $R(x)$; the number of possible values of r_1 cannot exceed $R(x^{\frac{1}{2}})$; and so on. The total number of values of q can therefore not exceed

$$S(x) = R(x)\, R(x^{\frac{1}{2}})\, R(x^{\frac{1}{4}}) \dots R(x^{2^{-\varpi}}), \tag{2.16}$$

where ϖ is the largest number such that

$$x^{2^{-\varpi}} \geqslant 2, \qquad x \geqslant 2^{2^{\varpi}}. \tag{2.161}$$

Thus

$$Q(x) \leqslant S(x). \tag{2.17}$$

2.2. We denote by f and g the largest numbers, such that

$$l_f \leqslant x, \tag{2.211}$$

$$l_1 l_2 \dots l_g \leqslant x. \tag{2.212}$$

It is known* (and may be proved by elementary methods) that constants A and B exist, such that

$$\vartheta(x) \geqslant Ax \quad (x \geqslant 2), \tag{2.221}$$

and

$$p_n \geqslant Bn \log n \quad (n \geqslant 1). \tag{2.222}$$

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* See Landau, *Handbuch*, pp. 79, 83, 214.

We have therefore $e^{Ap_f} \leqslant x$,

$$f \log f = O(\log x),$$

$$\log f = O(\log \log x); \tag{2.23}$$

and $$\sum_1^g \vartheta(p_\nu) \leqslant \log x, \qquad \sum_1^g p_\nu = O(\log x),$$

$$\sum_1^g \nu \log \nu = O(\log x), \qquad g^2 \log g = O(\log x),$$

$$g = O\sqrt{\left(\frac{\log x}{\log \log x}\right)}. \tag{2.24}$$

But it is easy to obtain an upper bound for $R(x)$ in terms of f and g. The number of numbers $l_1, l_2, \ldots$, not exceeding x, is not greater than f; the number of products, not exceeding x, of pairs of such numbers, is *a fortiori* not greater than $\frac{1}{2}f(f-1)$; and so on. Thus

$$R(x) \leqslant f + \frac{f(f-1)}{2!} + \frac{f(f-1)(f-2)}{3!} + \ldots,$$

where the summation need be extended to g terms only, since

$$l_1 l_2 \ldots l_g l_{g+1} > x.$$

A fortiori, we have

$$R(x) \leqslant 1 + f + \frac{f^2}{2!} + \ldots + \frac{f^g}{g!} < (1+f)^g = e^{g \log (1+f)}.$$

Thus

$$R(x) = e^{O(g \log f)} = e^{O\{\sqrt{(\log x \log \log x)}\}}, \tag{2.25}$$

by (2.23) and (2.24). Finally, since

$$\log \sqrt{x} \log \log \sqrt{x} < \tfrac{1}{2} \log x \log \log x,$$

it follows from (2.16) and (2.17) that

$$Q(x) = \exp\left[O\left\{\left(1 + \frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{2^\varpi}\right)\sqrt{(\log x \log \log x)}\right\}\right] \tag{2.26}$$

$$= e^{O\{\sqrt{(\log x \log \log x)}\}}.$$

2.3. A *lower* bound for $Q(x)$ may be found as follows. If g is defined as in 2.2, we have

$$l_1 l_2 \ldots l_g \leqslant x < l_1 l_2 \ldots l_g l_{g+1}.$$

It follows from the analysis of 2 . 2 that

$$l_{g+1} = e^{\vartheta(p_{g+1})} = e^{O(g \log g)},$$

and

$$l_1 l_2 \ldots l_g = \exp\left\{\sum_1^g \vartheta(p_\nu)\right\} = e^{O(g^2 \log g)}.$$

Thus

$$x < e^{O(g^2 \log g)};$$

which is only possible if g is greater than a constant positive multiple of

$$\sqrt{\left(\frac{\log x}{\log \log x}\right)}.$$

Now the numbers $l_1, l_2, \ldots, l_g$ can be combined in 2^g different ways, and each such combination gives a number q not greater than x. Thus

$$Q(x) \geqslant 2^g > \exp\left\{K\sqrt{\left(\frac{\log x}{\log \log x}\right)}\right\}, \tag{2 . 31}$$

where K is a positive constant. From (2 . 26) and (2 . 31) it follows that there are positive constants K and L such that

$$K\sqrt{\left(\frac{\log x}{\log \log x}\right)} < \log Q(x) < L\sqrt{(\log x \log \log x)}. \tag{2 . 32}$$

The inequalities (2 . 32) give a fairly accurate idea as to the order of magnitude of $Q(x)$. But they are much less precise than the inequalities (1 . 121). To obtain these requires the use of less elementary methods.

3.

The behaviour of $\mathfrak{Q}(s)$ when $s \to 0$ by positive values.

3 . 1. From the fact, already used in 2 . 1, that the class of numbers q is identical with the class of numbers of the form (2 . 12), it follows at once that

$$\mathfrak{Q}(s) = \Sigma \frac{1}{q^s} = \prod_1^\infty \left(\frac{1}{1 - l_n^{-s}}\right). \tag{1 . 14}$$

Both series and product are absolutely convergent for $\sigma > 0$, and

$$\log \mathfrak{Q}(s) = \phi(s) + \tfrac{1}{2}\phi(2s) + \tfrac{1}{3}\phi(3s) + \ldots, \tag{3 . 11}$$

where

$$\phi(s) = \sum_1^\infty l_n^{-s}. \tag{3 . 111}$$

We have also

$$(3.12)\quad \phi(s) = \frac{1-2^{-s}}{2^s-1} + 2^{-s}\frac{1-3^{-s}}{3^s-1} + 2^{-s}3^{-s}\frac{1-5^{-s}}{5^s-1} + \dots$$

$$= \frac{1}{2^s-1} + 2^{-s}\left(\frac{1}{3^s-1} - \frac{1}{2^s-1}\right) + 2^{-s}3^{-s}\left(\frac{1}{5^s-1} - \frac{1}{3^s-1}\right) + \dots$$

$$= \frac{1}{2^s-1} + \sum_1^\infty e^{-s\vartheta(p_n)} \int_{p_n}^{p_{n+1}} \frac{d}{dx}\left(\frac{1}{x^s-1}\right) dx$$

$$= \frac{1}{2^s-1} - s\int_2^\infty \frac{x^{s-1}}{(x^s-1)^2}\, e^{-s\vartheta(x)}\, dx.$$

3.2. LEMMA.—*If* $x > 1$, $s > 0$, *then*

$$(3.21)\qquad \frac{1}{(s\log x)^2} - \frac{1}{12} < \frac{x^s}{(x^s-1)^2} < \frac{1}{(s\log x)^2}.$$

Write $x^s = e^u$: then we have to prove that

$$(3.22)\qquad \frac{1}{u^2} - \frac{1}{12} < \frac{e^u}{(e^u-1)^2} < \frac{1}{u^2}$$

for all positive values of u; or (writing w for $\frac{1}{2}u$) that

$$(3.23)\qquad \frac{1}{w^2} - \frac{1}{3} < \frac{1}{\sinh^2 w} < \frac{1}{w^2}$$

for all positive values of w. But it is easy to prove that the function

$$f(w) = \frac{1}{w^2} - \frac{1}{\sinh^2 w}$$

is a steadily decreasing function of w, and that its limit when $w \to 0$ is $\frac{1}{3}$; and this establishes the truth of the lemma.

3.3. We have therefore

$$(3.31)\qquad \phi(s) = \frac{1}{2^s-1} - \phi_1(s) = \frac{1}{s\log 2} - \phi_1(s) + O(1),$$

where

$$(3.311)\quad \frac{1}{s}\int_2^\infty \left\{\frac{1}{(\log x)^2} - \frac{s^2}{12}\right\} e^{-s\vartheta(x)}\,\frac{dx}{x} < \phi_1(x) < \frac{1}{s}\int_2^\infty \frac{e^{-s\vartheta(x)}}{(\log x)^2}\,\frac{dx}{x}.$$

From the second of these inequalities, and (2 . 221), it follows that

$$\phi_1(s) < \frac{1}{s}\int_2^\infty \frac{e^{-Asx}}{(\log x)^2}\,\frac{dx}{x} = \frac{e^{-2As}}{s\log 2} - A\int_2^\infty \frac{e^{-Asx}}{\log x}\,dx$$

$$= \frac{1}{s\log 2} - A\int_2^\infty \frac{e^{-Asx}}{\log x}\,dx + O(1);$$

and so that

$$\phi(s) > A\int_2^\infty \frac{e^{-Asx}}{\log x}\,dx + O(1). \tag{3 . 32}$$

On the other hand there is a positive constant B, such that

$$\vartheta(x) < Bx \quad (x \geqslant 2).*$$

Thus

$$\phi_1(s) > \frac{1}{s}\int_2^\infty \left(\frac{1}{(\log x)^2} - \frac{s^2}{12}\right) e^{-Bsx}\,\frac{dx}{x} = \frac{1}{s}\int_2^\infty \frac{e^{-Bsx}}{(\log x)^2}\,\frac{dx}{x} + O(1)$$

$$= \frac{1}{s\log 2} - B\int_2^\infty \frac{e^{-Bsx}}{\log x}\,dx + O(1);$$

and so

$$\phi(s) < B\int_2^\infty \frac{e^{-Bsx}}{\log x}\,dx + O(1). \tag{3 . 33}$$

3 . 4. LEMMA.—*If H is any positive number, then*

$$J(s) = H\int_2^\infty \frac{e^{-Hsx}}{\log x}\,dx \sim \frac{1}{s\log(1/s)},$$

when $s \to 0$.

Given any positive number ϵ, we can choose ξ and X, so that

$$\int_0^\xi He^{-Hx}dx < \epsilon, \quad \int_X^\infty He^{-Hx}dx < \epsilon.$$

Now $$s\log\left(\frac{1}{s}\right) J(s) = \int_{2s}^\infty \frac{He^{-Hu}\log(1/s)}{\log u + \log(1/s)}\,du = \int_{2s}^{\sqrt{s}} + \int_{\sqrt{s}}^\xi + \int_\xi^X + \int_X^\infty$$

$$= j_1(s) + j_2(s) + j_3(s) + j_4(s),$$

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* Landau, *Handbuch*, *l.c.*

say. And we have

$$0 < j_1(s) < \frac{\log(1/s)}{\log 2} \int_{2s}^{\sqrt{s}} He^{-Hu}\,du = O\{\sqrt{s}\log(1/s)\} = o(1),$$

$$0 < j_2(s) < 2\int_0^{\xi} He^{-Hu}\,du < 2\epsilon,$$

$$j_3(s) = \int_{\xi}^{X} He^{-Hu}\,du + o(1),$$

$$0 < j_4(s) < \int_X^{\infty} He^{-Hu}\,du < \epsilon;$$

and so $$\left|1 - s\log\left(\frac{1}{s}\right) J(s)\right| = \left|\int_0^{\infty} He^{-Hu}\,du - j_1(s) - j_2(s) - j_3(s) - j_4(s)\right|$$

$$< 5\epsilon + o(1) < 6\epsilon,$$

for all sufficiently small values of s.

3 . 5. From (3 . 32), (3 . 33), and the lemma just proved, it follows that

$$\phi(s) = \Sigma\, l_n^{-s} \sim \frac{1}{s\log(1/s)}. \tag{3 . 51}$$

From this formula we can deduce an asymptotic formula for $\log \mathfrak{Q}(s)$. We choose N so that

$$\sum_{N<n} \frac{1}{n^2} < \epsilon, \tag{3 . 52}$$

and we write

$$\log \mathfrak{Q}(s) = \Sigma \frac{1}{n}\,\phi(ns) = \sum_{1 \leqq n \leqq N} + \sum_{N<n<1/\sqrt{s}} + \sum_{1/\sqrt{s} \leqq n \leqq 1/s} + \sum_{1/s<n} \tag{3 . 53}$$

$$= \Phi_1(s) + \Phi_2(s) + \Phi_3(s) + \Phi_4(s),$$

say.

In the first place

$$\Phi_1(s) = \frac{1+o(1)}{s\log(1/s)} \sum_1^N \frac{1}{n^2}. \tag{3 . 541}$$

In the second place

$$\phi(ns) = O\left\{\frac{1}{ns\log(1/ns)}\right\},$$

and

$$\log(1/ns) > \tfrac{1}{2}\log(1/s),$$

if $N < n < 1/\sqrt{s}$. It follows that a constant K exists such that

$$\Phi_2(s) < \frac{K}{s \log(1/s)} \sum_{N<n} \frac{1}{n^2} < \frac{K\epsilon}{s \log(1/s)}. \tag{3.542}$$

Thirdly, $\sqrt{s} \leqslant ns \leqslant 1$ in $\Phi_3(s)$, and a constant L exists such that

$$\phi(ns) < \frac{L}{\sqrt{s} \log(1/s)}.$$

Thus

$$\Phi_3(s) < \frac{L}{\sqrt{s} \log(1/s)} \sum_{1}^{1/s} \frac{1}{n} < \frac{2L}{\sqrt{s}}, \tag{3.543}$$

for all sufficiently small values of s.

Finally, in $\Phi_4(s)$ we have $ns > 1$, and a constant M exists such that

$$\phi(ns) < M 2^{-ns}.$$

Thus

$$\Phi_4(s) < M \sum_{1/s<n} \frac{2^{-ns}}{n} < sM \sum_{1/s<n} 2^{-ns} < \frac{sM}{1-2^{-s}} = O(1). \tag{3.544}$$

From (3.53), (3.541)–(3.544), and (3.52) it follows that

$$\log \mathfrak{A}(s) = \frac{1}{s \log(1/s)} \left[\{1+o(1)\} \left(\frac{\pi^2}{6} + \rho \right) + \rho' \right] + O\left\{ \frac{1}{\sqrt{s} \log(1/s)} \right\} + O(1), \tag{3.55}$$

where

$$|\rho| < \epsilon, \quad |\rho'| < K\epsilon.$$

Thus

$$\log \mathfrak{A}(s) \sim \frac{\pi^2}{6s \log(1/s)}, \tag{3.56}$$

or

$$\mathfrak{A}(s) = \exp\left[\{1+o(1)\} \frac{\pi^2}{6s \log(1/s)} \right]. \tag{3.57}$$

4.

A Tauberian theorem.

4.1. The passage from (3.57) to (1.12) depends upon a theorem of the "Tauberian" type.

THEOREM A.—*Suppose that*

(1) $\lambda_1 \geqslant 0, \quad \lambda_n > \lambda_{n-1}, \quad \lambda_n \to \infty$;

(2) $\lambda_n/\lambda_{n-1} \to 1$;

(3) $a_n \geqslant 0$;

(4) $A > 0, \quad \alpha > 0$;

(5) $\Sigma a_n e^{-\lambda_n s}$ *is convergent for* $s > 0$;

(6) $$f(s) = \Sigma a_n e^{-\lambda_n s} = \exp\left[\{1+o(1)\} A s^{-\alpha} \left\{\log\left(\frac{1}{s}\right)\right\}^{-\beta}\right],$$

when $s \to 0$. *Then*

$$A_n = a_1 + a_2 + \ldots + a_n = \exp\left[\{1+o(1)\} B \lambda_n^{\alpha/(1+\alpha)} (\log \lambda_n)^{-\beta/(1+\alpha)}\right],$$

where
$$B = A^{1/(1+\alpha)} \alpha^{-\alpha/(1+\alpha)} (1+\alpha)^{1+[\beta/(1+\alpha)]},$$

when $n \to \infty$.

We are given that

$$(4.11) \quad (1-\delta) A s^{-\alpha} \left(\log\frac{1}{s}\right)^{-\beta} < \log f(s) < (1+\delta) A s^{-\alpha} \left(\log\frac{1}{s}\right)^{-\beta},$$

for every positive δ and all sufficiently small values of s; and we have to show that

$$(4.12) \quad (1-\epsilon) B \lambda_n^{\alpha/(1+\alpha)} (\log \lambda_n)^{-\beta/(1+\alpha)} < \log A_n < (1+\epsilon) B \lambda_n^{\alpha/(1+\alpha)} (\log \lambda_n)^{-\beta/(1+\alpha)},$$

for every positive ϵ and all sufficiently large values of n.

In the argument which follows we shall be dealing with three variables, δ, s, and n (or m), the two latter variables being connected by an equation or by inequalities, and with an auxiliary parameter ξ. We shall use the letter η, without a suffix, to denote generally a function of δ, s, and n (or m)*, which is not the same in different formulæ, but in all cases tends to zero when δ and s tend to zero and n (or m) to infinity; so that, given any positive ϵ, we have

$$0 < |\eta| < \epsilon,$$

for
$$0 < \delta < \delta_0, \quad 0 < s < s_0, \quad n > n_0.$$

* η may, of course, in some cases be a function of some of these variables only.

We shall use the symbol η_ζ to denote a function *of* ζ *only* which tends to zero with ζ, so that

$$0 < |\eta_\zeta| < \epsilon,$$

if ζ is small enough. It is to be understood that the choice of a ζ to satisfy certain conditions is in all cases prior to that of δ, s, and n (or m). Finally, we use the letters H, K, ... to denote positive numbers independent of these variables *and of* ζ.

The *second* of the inequalities (4 . 12) is very easily proved. For

$$(4.131)\qquad A_n e^{-\lambda_n s} < a_1 e^{-\lambda_1 s} + a_2 e^{-\lambda_2 s} + \ldots + a_n e^{-\lambda_n s}$$

$$< f(s) < \exp\left\{(1+\delta) A s^{-\alpha} \left(\log \frac{1}{s}\right)^{-\beta}\right\},$$

$$(4.132)\qquad A_n < \exp \chi,$$

where

$$(4.1321)\qquad \chi = (1+\delta) A s^{-\alpha} \left(\log \frac{1}{s}\right)^{-\beta} + \lambda_n s.$$

We can choose a value of s, corresponding to every large value of n, such that

$$(4.14)\quad (1-\delta) A\alpha s^{-1-\alpha} \left(\log \frac{1}{s}\right)^{-\beta} < \lambda_n < (1+\delta) A\alpha s^{-1-\alpha} \left(\log \frac{1}{s}\right)^{-\beta}.$$

From these inequalities we deduce, by an elementary process of approximation,

$$(4.151)\quad (1-\eta)(A\alpha)^{-1/(1+\alpha)} \lambda_n^{1/(1+\alpha)} \left(\log \frac{1}{s}\right)^{\beta/(1+\alpha)}$$

$$< \frac{1}{s} < (1+\eta)(A\alpha)^{-1/(1+\alpha)} \lambda_n^{1/(1+\alpha)} \left(\log \frac{1}{s}\right)^{\beta/(1+\alpha)},$$

$$(4.152)\qquad \frac{1-\eta}{1+\alpha} \log \lambda_n < \log \frac{1}{s} < \frac{1+\eta}{1+\alpha} \log \lambda_n,$$

$$(4.153)\quad (1-\eta) \frac{\alpha B}{1+\alpha} \lambda_n^{-1/(1+\alpha)} (\log \lambda_n)^{-\beta/(1+\alpha)}$$

$$< s < (1+\eta) \frac{\alpha B}{1+\alpha} \lambda_n^{-1/(1+\alpha)} (\log \lambda_n)^{-\beta/(1+\alpha)},$$

$$(4.154)\qquad \chi < (1+\eta) B \lambda_n^{\alpha/(1+\alpha)} (\log \lambda_n)^{-\beta/(1+\alpha)}$$

We have therefore

$$\log A_n < (1+\epsilon) B \lambda_n^{\alpha/(1+\alpha)} (\log \lambda_n)^{-\beta/(1+\alpha)}, \tag{4.16}$$

for every positive ϵ and all sufficiently large values of n.*

4.2. We have

$$\begin{aligned} f(s) = \Sigma a_n e^{-\lambda_n s} &= \Sigma A_n (e^{-\lambda_n s} - e^{-\lambda_{n+1} s}) \\ &= s \sum_1^\infty A_n \int_{\lambda_n}^{\lambda_{n+1}} e^{-sx} dx = s \int_0^\infty \boldsymbol{A}(x) e^{-sx} dx, \end{aligned} \tag{4.21}$$

where $\boldsymbol{A}(x)$ is the discontinuous function defined by

$$\boldsymbol{A}(x) = A_n \quad (\lambda_n \leqslant x < \lambda_{n+1}),\dagger$$

so that, by (4.16),

$$\log \boldsymbol{A}(x) < (1+\epsilon) B x^{\alpha/(1+\alpha)} (\log x)^{-\beta/(1+\alpha)} \tag{4.22}$$

for every positive ϵ and all sufficiently large values of x.

We have therefore

$$\begin{aligned} \exp\left\{(1-\delta) A s^{-\alpha} \left(\log \frac{1}{s}\right)^{-\beta}\right\} &< s \int_0^\infty \boldsymbol{A}(x) e^{-sx} dx \\ &< \exp\left\{(1+\delta) A s^{-\alpha} \left(\log \frac{1}{s}\right)^{-\beta}\right\} \end{aligned} \tag{4.23}$$

for every positive δ and all sufficiently small values of s.

We define λ_x, a steadily increasing and continuous function of the continuous variable x, by the equation

$$\lambda_x = \lambda_n + (x-n)(\lambda_{n+1} - \lambda_n) \quad (n \leqslant x \leqslant n+1).$$

We can then choose m so that

$$\frac{1}{s} = \frac{1+\alpha}{\alpha B} \lambda_m^{1/(1+\alpha)} (\log \lambda_m)^{\beta/(1+\alpha)}. \tag{4.24}$$

We shall now show that the limits of the integral in (4.23) may be re-

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* We use the second inequality (4.12) in the proof of the first. It would be sufficient for our purpose to begin by proving a result cruder than (4.16), with any constant K on the right-hand side instead of $(1+\epsilon)B$. But it is equally easy to obtain the more precise inequality. Compare the argument in the second of the two papers by Hardy and Littlewood quoted on p. 114 (pp. 143 *et seq.*).

† Compare Hardy and Riesz, "The General Theory of Dirichlet's Series", *Cambridge Tracts in Mathematics*, No. 18, 1915, p. 24.

placed by $(1-\xi)\lambda_m$ and $(1+\xi)\lambda_m$, where ξ is an arbitrary positive number less than unity.

We write

(4.25)

$$J(s) = s\int_0^\infty \boldsymbol{A}(x)e^{-sx}dx = s\left\{\int_0^{\lambda_m/H} + \int_{\lambda_m/H}^{(1-\xi)\lambda_m} + \int_{(1-\xi)\lambda_m}^{(1+\xi)\lambda_m} + \int_{(1+\xi)\lambda_m}^{H\lambda_m} + \int_{H\lambda_m}^{\infty}\right\}$$

$$= J_1+J_2+J_3+J_4+J_5,$$

where H is a constant, in any case greater than 1, and large enough to satisfy certain further conditions which will appear in a moment; and we proceed to show that J_1, J_2, J_4, and J_5 are negligible in comparison with the exponentials which occur in (4.23), and so in comparison with J_3.

4.3. The integrals J_1 and J_5 are easily disposed of. In the first place we have

(4.31)
$$J_1 = s\int_0^{\lambda_m/H} \boldsymbol{A}(x)e^{-sx}dx < \boldsymbol{A}\left(\frac{\lambda_m}{H}\right)$$

$$< \exp\left\{(1+\delta)B\left(\frac{\lambda_m}{H}\right)^{\alpha/(1+\alpha)}\left(\log\frac{\lambda_m}{H}\right)^{-\beta/(1+\alpha)}\right\},$$

by (4.22).* It will be found, by a straightforward calculation, that this expression is less than

(4.32)
$$\exp\left\{(1+\eta)A(1+\alpha)H^{-\alpha/(1+\alpha)}s^{-\alpha}\left(\log\frac{1}{s}\right)^{-\beta}\right\},$$

and is therefore certainly negligible if H is sufficiently large.

Thus J_1 is negligible. To prove that J_5 is negligible we prove first that

$$sx > 4Bx^{\alpha/(1+\alpha)}(\log x)^{-\beta/(1+\alpha)},$$

if $s > H\lambda_m$ and H is large enough†. It follows that

$$J_5 = s\int_{H\lambda_m}^{\infty} \boldsymbol{A}(x)e^{-sx}dx < s\int_{H\lambda_m}^{\infty} \exp\left\{(1+\delta)Bx^{\alpha/(1+\alpha)}(\log x)^{-\beta/(1+\alpha)} - sx\right\}dx$$

$$< s\int_0^\infty e^{-\frac{1}{2}sx}dx = \tfrac{1}{2},$$

and is therefore negligible.

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* With δ in the place of ϵ.

† We suppress the details of the calculation, which is quite straightforward.

4.4. The integrals J_2 and J_4 may be discussed in practically the same way, and we may confine ourselves to the latter.

We have

$$J_4(s) = s\int_{(1+\zeta)\lambda_m}^{H\lambda_m} A(x)\, e^{-sx}\, dx < s\int_{(1+\zeta)\lambda_m}^{H\lambda_m} e^{\psi}\, dx, \tag{4.41}$$

where

$$\psi = (1+\delta)\, B x^{\alpha/(1+\alpha)} (\log x)^{-\beta/(1+\alpha)} - sx. \tag{4.411}$$

The maximum of the function ψ occurs for $x = x_0$, where

$$\frac{1}{s} = (1+\eta)\,\frac{1+\alpha}{\alpha B}\, x_0^{1/(1+\alpha)} (\log x_0)^{\beta/(1+\alpha)}. \tag{4.42}$$

From this equation, and (4.24), it plainly results that

$$(1-\eta)\lambda_m < x_0 < (1+\eta)\lambda_m, \tag{4.43}$$

and that x_0 falls (when δ and s are small enough) between $(1-\zeta)\lambda_m$ and $(1+\zeta)\lambda_m$.

Let us write

$$x = x_0 + \xi$$

in J_4. Then

$$\psi(x) = \psi(x_0) + \tfrac{1}{2}(1+\delta) B \xi^2 \frac{d^2}{dx_1^2}\left\{x_1^{\alpha/(1+\alpha)} (\log x_1)^{-\beta/(1+\alpha)}\right\},$$

where $x_0 < x_1 < x$ and *a fortiori*

$$(1-\zeta)\lambda_m < x_1 < H\lambda_m.$$

It follows that

$$\frac{d^2}{dx_1^2}\left\{x_1^{\alpha/(1+\alpha)} (\log x_1)^{-\beta/(1+\alpha)}\right\} < -K\lambda_m^{\alpha/(1+\alpha)-2} (\log \lambda_m)^{-\beta/(1+\alpha)}. \tag{4.44}$$

On the other hand, an easy calculation shows that

$$(1-\eta) A s^{-\alpha} \left(\log \frac{1}{s}\right)^{-\beta} < \psi(x_0) < (1+\eta) A s^{-\alpha} \left(\log \frac{1}{s}\right)^{-\beta}. \tag{4.45}$$

Thus

$$\begin{aligned} J_4 &< \exp\left\{(1+\eta) A s^{-\alpha}\left(\log \frac{1}{s}\right)^{-\beta}\right\} \\ &\qquad \times \int_{(\zeta-\eta)\lambda_m}^{\infty} \exp\left\{-L\xi^2 \lambda_m^{\alpha/(1+\alpha)-2} (\log \lambda_m)^{-\beta/(1+\alpha)}\right\} d\xi \\ &< \exp\left\{(1+\eta) A s^{-\alpha}\left(\log \frac{1}{s}\right)^{-\beta} - M\xi^2 \lambda_m^{\alpha/(1+\alpha)} (\log \lambda_m)^{-\beta/(1+\alpha)}\right\} \\ &< \exp\left\{(1+\eta-N\xi^2) A s^{-\alpha}\left(\log \frac{1}{s}\right)^{-\beta}\right\}. \end{aligned} \tag{4.46}$$

Since ξ is independent of δ and s, this inequality shows that J_4 is negligible; and a similar argument may be applied to J_2.

4.5. We may therefore replace the inequalities (4.23) by

$$(4.51)\quad \exp\left\{(1-\delta)As^{-\alpha}\left(\log\frac{1}{s}\right)^{-\beta}\right\}$$

$$< s\int_{(1-\zeta)\lambda_m}^{(1+\zeta)\lambda_m} \boldsymbol{A}(x)e^{-sx}\,dx < \exp\left\{(1+\delta)As^{-\alpha}\left(\log\frac{1}{s}\right)^{-\beta}\right\}.$$

Since $\boldsymbol{A}(x)$ is a steadily increasing function of x, it follows that

$$(4.521)\quad \exp\left\{(1-\delta)As^{-\alpha}\left(\log\frac{1}{s}\right)^{-\beta}\right\} < s\boldsymbol{A}\{(1+\xi)\lambda_m\}\int_{(1-\zeta)\lambda_m}^{(1+\zeta)\lambda_m} e^{-sx}\,dx,$$

$$(4.522)\quad \exp\left\{(1+\delta)As^{-\alpha}\left(\log\frac{1}{s}\right)^{-\beta}\right\} > s\boldsymbol{A}\{(1-\xi)\lambda_m\}\int_{(1-\zeta)\lambda_m}^{(1+\zeta)\lambda_m} e^{-sx}\,dx;$$

or

$$(4.531)\quad (e^{\zeta s\lambda_m}-e^{-\zeta s\lambda_m})\boldsymbol{A}\{(1-\xi)\lambda_m\} < \exp\left\{(1+\delta)As^{-\alpha}\left(\log\frac{1}{s}\right)^{-\beta}+\lambda_m s\right\},$$

$$(4.532)\quad (e^{\zeta s\lambda_m}-e^{-\zeta s\lambda_m})\boldsymbol{A}\{(1+\xi)\lambda_m\} > \exp\left\{(1-\delta)As^{-\alpha}\left(\log\frac{1}{s}\right)^{-\beta}+\lambda_m s\right\}.$$

If we substitute for s, in terms of λ_m, in the right-hand sides of (4.531) and (4.532), we obtain expressions of the form

$$\exp\{(1+\eta)B\lambda_m^{\alpha/(1+\alpha)}(\log\lambda_m)^{-\beta/(1+\alpha)}\}.$$

On the other hand $\qquad e^{\zeta s\lambda_m}-e^{-\zeta s\lambda_m}$

is of the form $\qquad \exp\{\eta_\zeta\lambda_m^{\alpha/(1+\alpha)}(\log\lambda_m)^{-\beta/(1+\alpha)}\}.$

We have thus

$$(4.541)\quad A\{(1-\xi)\lambda_m\} < \exp\{(1+\eta_\zeta+\eta)B\lambda_m^{\alpha/(1+\alpha)}(\log\lambda_m)^{-\beta/(1+\alpha)}\},$$

$$(4.542)\quad A\{(1+\xi)\lambda_m\} > \exp\{(1-\eta_\zeta-\eta)B\lambda_m^{\alpha/(1+\alpha)}(\log\lambda_m)^{-\beta/(1+\alpha)}\}.$$

Now let ν be any number such that

$$(4.55)\qquad (1-\xi)\lambda_m \leqslant \lambda_\nu \leqslant (1+\xi)\lambda_m.$$

Since $\lambda_n/\lambda_{n-1}\to 1$, it is clear that *all* numbers n from a certain point onwards will fall among the numbers ν. It follows from (4.541) and

(4 . 542) that

$$(4.56)\quad \exp\{(1-\eta_\zeta-\eta)(1-\eta_\zeta)B\lambda_\nu^{\alpha/(1+\alpha)}(\log\lambda_\nu)^{-\beta/(1+\alpha)}\} < A(\lambda_\nu) < \exp\{(1+\eta_\zeta+\eta)(1+\eta_\zeta)B\lambda_\nu^{\alpha/(1+\alpha)}(\log\lambda_\nu)^{-\beta/(1+\alpha)}\};$$

and therefore that, given ϵ, we can choose first ζ and then n_0 so that

$$(4.57)\quad \exp\{(1-\epsilon)B\lambda_n^{\alpha/(1+\alpha)}(\log\lambda_n)^{-\beta/(1+\alpha)} < A(\lambda_n) < \exp\{(1+\epsilon)B\lambda_n^{\alpha/(1+\alpha)}(\log\lambda_n)^{-\beta/(1+\alpha)}\},$$

for $n \geqslant n_0$. This completes the proof of the theorem.

4 . 6. There is of course a corresponding "Abelian" theorem, which we content ourselves with enunciating. This theorem is naturally not limited by the restriction that the coefficients a_n are positive.

THEOREM B.—*Suppose that*

(1) $\lambda_1 \geqslant 0$, $\lambda_n > \lambda_{n-1}$, $\lambda_n \to \infty$;

(2) $\lambda_n/\lambda_{n-1} \to 1$;

(3) $A > 0$, $0 < \alpha < 1$;

(4) $A_n = a_1+a_2+\ldots+a_n = \exp[\{(1+o(1)\}A\lambda_n^\alpha(\log\lambda_n)^{-\beta}]$,

when $n \to \infty$. *Then the series* $\Sigma a_n e^{-\lambda_n s}$ *is convergent for* $s > 0$, *and*

$$f(s) = \Sigma a_n e^{-\lambda_n s} = \exp\left[\{1+o(1)\}Bs^{-\alpha/(1-\alpha)}\left(\log\frac{1}{s}\right)^{-\beta/(1-\alpha)}\right],$$

where

$$B = A^{1/(1-\alpha)}\alpha^{\alpha/(1-\alpha)}(1-\alpha)^{1+[\beta/(1-\alpha)]}],$$

when $s \to 0$.

The proof of this theorem, which is naturally easier than that of the correlative Tauberian theorem, should present no difficulty to anyone who has followed the analysis which precedes.

4 . 7. The simplest and most interesting cases of Theorems A and B are those in which

$$\lambda_n = n, \quad \beta = 0.$$

It is then convenient to write x for e^{-s}. We thus obtain

THEOREM C.—*If* $A > 0$, $0 < \alpha < 1$, *and*

$$\log A_n = \log (a_1 + a_2 + \ldots + a_n) \sim A n^{\alpha},$$

then the series $\Sigma a_n x^n$ *is convergent for* $|x| < 1$, *and*

$$\log f(x) = \log (\Sigma a_n x^n) \sim B(1-x)^{-\alpha/(1-\alpha)},$$

where

$$B = (1-\alpha)\,\alpha^{\alpha/(1-\alpha)} A^{1/(1-\alpha)},$$

when $x \to 1$ *by real values.*

If the coefficients are positive the converse inference is also correct. That is to say, if

$$A > 0, \quad \alpha > 0,$$

and

$$\log f(x) \sim A(1-x)^{-\alpha},$$

then

$$\log A_n \sim B n^{\alpha/(1+\alpha)},$$

where

$$B = (1+\alpha)\,\alpha^{-\alpha/(1+\alpha)} A^{1/(1+\alpha)}.$$

5.

Application to our problem, and to other problems in the Theory of Numbers.

5 . 1. We proved in 3 that

$$\log \mathfrak{Q}(s) \sim \frac{\pi^2}{6s \log (1/s)}. \tag{3 . 56}$$

In Theorem A take

$$\lambda_n = \log n, \quad A = \frac{\pi^2}{6}, \quad \alpha = 1, \quad \beta = 1.$$

Then all the conditions of the theorem are satisfied. And A_n is $Q(n)$, the number of numbers q not exceeding n. We have therefore

$$\log Q(n) \sim B \sqrt{\left(\frac{\log n}{\log \log n}\right)}, \tag{5 . 11}$$

where

$$B = 2^{\frac{3}{2}} \sqrt{\left(\frac{\pi^2}{6}\right)} = \frac{2\pi}{\sqrt{3}}. \tag{5 . 12}$$

5 . 2. The method which we have followed in solving this problem is one capable of many other interesting applications.

Suppose, for example, that $R_r(n)$ is the number of ways in which n can be represented as the sum of any number of r-th powers of positive integers.* We shall prove that

$$(5\cdot21)\qquad \log R_r(n) \sim (r+1)\left\{\frac{1}{r}\Gamma\left(\frac{1}{r}+1\right)\zeta\left(\frac{1}{r}+1\right)\right\}^{r/(r+1)} n^{1/(r+1)}.$$

In particular, if $P(n) = R_1(n)$ is the number of partitions of n, then

$$(5\cdot22)\qquad \log P(n) \sim \pi\sqrt{\left(\frac{2n}{3}\right)}.$$

We need only sketch the proof, which is in principle similar to the main proof of this paper. We have

$$\sum_1^\infty R_r(n)e^{-ns} = \prod_1^\infty\left(\frac{1}{1-e^{-sv^r}}\right),$$

and so

$$(5\cdot23)\qquad f(s) = \sum_1^\infty\{R_r(n)-R_r(n-1)\}e^{-ns} = \prod_2^\infty\left(\frac{1}{1-e^{-sv^r}}\right).\dagger$$

It is obvious that $R_r(n)$ increases with n and that all the coefficients in $f(s)$ are positive. Again,

$$(5\cdot24)\qquad \log f(s) = \sum_2^\infty \log\left(\frac{1}{1-e^{-sv^r}}\right) = \sum_2^\infty (e^{-sv^r}+\tfrac{1}{2}e^{-2sv^r}+\ldots)$$

$$= \sum_{k=1}^\infty \frac{1}{k}\phi(ks),$$

where

$$(5\cdot241)\qquad \phi(s) = \sum_2^\infty e^{-sv^r}.$$

But

$$(5\cdot25)\qquad \phi(s) \sim \Gamma\left(\frac{1}{r}+1\right)s^{-1/r},$$

when $s\to 0$; and we can deduce, by an argument similar to that of 3·5, that

$$(5\cdot26)\qquad \log f(s) \sim \Gamma\left(\frac{1}{r}+1\right)\zeta\left(\frac{1}{r}+1\right)s^{-1/r}.$$

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* Thus $28 = 3^3+1^3 = 3.2^3+4.1^3 = 2.2^3+12.1^3 = 2^3+20.1^3 = 28.1^3$:

and $R_3(28) = 5$.

The *order* of the powers is supposed to be indifferent, so that (*e.g.*) 3^3+1^3 and 1^3+3^3 are not reckoned as separate representations.

† $R_r(0)$ is to be interpreted as zero.

We now obtain (5 . 21) by an application of Theorem A, taking

$$\lambda_n = n, \quad \alpha = \frac{1}{r}, \quad \beta = 0, \quad A = \Gamma\left(\frac{1}{r}+1\right)\zeta\left(\frac{1}{r}+1\right).$$

In a similar manner we can show that, if $S(n)$ is the number of partitions of n into *different* positive integers, so that

$$\Sigma S(n)\, e^{-ns} = (1+e^{-s})(1+e^{-2s})(1+e^{-3s})+\ldots$$

$$= \frac{1}{(1-e^{-s})(1-e^{-3s})(1-e^{-5s})\ldots},$$

then

$$\log S(n) \sim \pi\sqrt{\left(\frac{n}{3}\right)}; \tag{5 . 27}$$

that if $T_r(n)$ is the number of representations of n as the sum of r-th powers of *primes*, then

$$\log T_r(n) \sim (r+1)\left\{\Gamma\left(\frac{1}{r}+2\right)\zeta\left(\frac{1}{r}+1\right)\right\}^{r/(r+1)} n^{1/(r+1)}(\log n)^{-r/(r+1)}; \tag{5 . 28}$$

and, in particular, that if $T(n) = T_1(n)$ is the number of partitions of n into primes, then

$$\log T(n) \sim \frac{2\pi}{\sqrt{3}}\sqrt{\left(\frac{n}{\log n}\right)}. \tag{5 . 281}$$

Finally, we can show that if r and s are positive integers, $a > 0$, and $0 \leqslant b \leqslant 1$, and

$$\Sigma\phi(n)\, x^n = \frac{\{(1+ax)(1+ax^2)(1+ax^3)\ldots\}^r}{\{(1-bx)(1-bx^2)(1-bx^3)\ldots\}^s}, \tag{5 . 291}$$

then

$$\log\phi(n) \sim 2\sqrt{(cn)}, \tag{5 . 292}$$

where

$$c = r\int_0^a \frac{\log(1+t)}{t}\,dt - s\int_0^b \frac{\log(1-t)}{t}\,dt. \tag{5 . 2921}$$

In particular, if $a = 1$, $b = 1$, and $r = s$, we have

$$\Sigma\phi(n)\, x^n = (1-2x+2x^4-2x^9+\ldots)^{-r}, \tag{5 . 293}$$

$$\log\phi(n) \sim \pi\sqrt{(rn)}. \tag{5 . 294}$$

[*Added March 28th*, 1917.—Since this paper was written M. G. Valiron ("Sur la croissance du module maximum des séries entières",

Bulletin de la Société mathématique de France, Vol. 44, 1916, pp. 45–64) has published a number of very interesting theorems concerning power-series which are more or less directly related to ours. M. Valiron considers power-series only, and his point of view is different from ours, in some respects more restricted and in others more general.

He proves in particular that *the necessary and sufficient conditions that*

$$\log M(r) \sim \frac{A}{(1-r)^{\alpha}},$$

where $M(r)$ *is the maximum modulus of* $f(x) = \Sigma a_n x^n$ *for* $|x| = r$, *are that*

$$\log |a_n| < (1+\epsilon)(1+\alpha) A^{1/(1+\alpha)} \left(\frac{n}{\alpha}\right)^{\alpha/(1+\alpha)}$$

for $n > n_0(\epsilon)$, *and*

$$\log |a_n| > (1-\epsilon_p)(1+\alpha) A^{1/(1+\alpha)} \left(\frac{n}{\alpha}\right)^{\alpha/(1+\alpha)}$$

for $n = n_p$ $(p = 1, 2, 3, \ldots)$, *where* $n_{p+1}/n_p \to 1$ *and* $\epsilon_p \to 0$ *as* $p \to \infty$.

M. Valiron refers to previous, but less general or less precise, results given by Borel (*Leçons sur les séries à termes positifs*, 1902, Ch. 5) and by Wiman ("Uber dem Zusammenhang zwischen dem Maximal-betrage einer analytischen Funktion und dem grössten Gliede der zugehörigen Taylor'schen Reihe", *Acta Mathematica*, Vol. 37, 1914, pp. 305–326). We may add a reference to Le Roy, "Valeurs asymptotiques de certaines séries procédant suivant les puissances entières et positives d'une variable réelle", *Bulletin des sciences mathématiques*, Ser. 2, Vol. 24, 1900, pp. 245–268.

We have more recently obtained results concerning $P(n)$, the number of partitions of n, far more precise than (5 . 22). A preliminary account of these researches has appeared, under the title "Une formule asymptotique pour le nombre des partitions de n", in the *Comptes Rendus* of January 2nd, 1917; and a fuller account has been presented to the Society. See *Records of Proceedings at Meetings*, March 1st, 1917.]

THE RELATIONS BETWEEN SPHERICAL, CYLINDRICAL, AND SPHEROIDAL HARMONICS

By G. B. JEFFERY.

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IN a paper published in the *Mathematische Annalen** in 1903, Whittaker has shown that a general solution of Laplace's equation may be written in the form

$$\int_0^{2\pi} f(x\cos t+y\sin t+iz,\ t)\,dt.$$

In the same paper he gave the integrals of this type corresponding to spherical and cylindrical harmonics. With the usual notation

$$r^n P_n^m(\cos\theta)\begin{matrix}\cos\\ \sin\end{matrix} m\phi = \frac{i^{m-n}}{2\pi}\frac{\Gamma(n+m+1)}{\Gamma(n+1)}\int_0^{2\pi}(x\cos t+y\sin t+iz)^n\begin{matrix}\cos\\ \sin\end{matrix} mt\,dt, \quad (1)$$

and

$$e^{-kz}J_m(k\varpi)\begin{matrix}\cos\\ \sin\end{matrix} m\phi = \frac{i^{-m}}{2\pi}\int_0^{2\pi} e^{ik(x\cos t+y\sin t+iz)}\begin{matrix}\cos\\ \sin\end{matrix} mt\,dt. \quad (2)$$

More recently E. Blades† has shown that interior spheroidal harmonics are given by

$$P_n^m(i\xi)\,P_n^m(\cos\eta)\begin{matrix}\cos\\ \sin\end{matrix} m\phi = \frac{1}{2\pi}\frac{(n+m)!}{(n-m)!}\int_0^{2\pi} P_n\left(\frac{x\cos t+y\sin t+iz}{c}\right)\begin{matrix}\cos\\ \sin\end{matrix} mt\,dt, \quad (3)$$

where the coordinates are defined by

$$x = c\sqrt{\xi^2+1}\sin\eta\cos\phi,\quad y = c\sqrt{\xi^2+1}\sin\eta\sin\phi,\quad z = c\xi\cos\eta, \quad (4)$$

while the author has shown‡ that the corresponding harmonics which vanish at infinity are given by

$$Q_n^m(i\xi)\,P_n^m(\cos\eta)\begin{matrix}\cos\\ \sin\end{matrix} m\phi = \frac{1}{2\pi}\frac{(n+m)!}{(n-m)!}\int_0^{2\pi} Q_n\left(\frac{x\cos t+y\sin t+iz}{c}\right)\begin{matrix}\cos\\ \sin\end{matrix} mt\,dt. \quad (5)$$

* Vol. LVII, p. 333.

† *Proc. Edinburgh Mathematical Society*, Vol. XXXIII, 1915.

‡ *Ibid.*

Without examining the extreme range of the values of the variables for which the formulæ (1), (2), (3), (5), are valid, it will be sufficient for our purpose to note that they all hold good if $z > |c|$.

The object of the present paper is to show that Whittaker's general solution provides a ready means of expressing a given potential function in terms of different harmonics. Thus if we can expand the function f in a series of powers of its first argument, we can express the potential in spherical harmonics; if we expand in a Fourier series we have the same potential expressed in terms of Bessel functions; while if we desire to use spheroidal harmonics we must expand f in a series of Legendre functions.

We shall find it convenient to write

$$x \cos t + y \sin t + iz \equiv \xi.$$

To transform spherical to spheroidal harmonics, having the same axis and origin.—Let c be the parameter of the spheroidal coordinates. If c is pure imaginary the coordinate surfaces $\xi = \text{const.}$ are confocal oblate spheroids; if c is real they are prolate spheroids having a common focal circle of radius $|c|$. We will confine our investigation to spherical harmonics of integral degree. Those of positive degree are included in the set

$$r^n P_n^m(\cos\theta)\cos m\phi = \frac{i^{m-n}}{2\pi}\,\frac{(n+m)!}{n!}\int_0^{2\pi} \xi^n \cos mt\,dt,$$

it being understood that throughout the paper $\cos m\phi$ may be replaced by $\sin m\phi$, if at the same time $\cos mt$ is replaced by $\sin mt$. By a well known result (Ferrers, *Spherical Harmonics*, p. 27)

$$\left(\frac{\xi}{c}\right)^n = n! \sum_{p=0}^{q} \frac{(2n-4p+1)(n-p)!}{(2n-2p+1)!\,p!}\, 2^{n-2p} P_{n-2p}\left(\frac{\xi}{c}\right),$$

where q is $\frac{1}{2}n$ or $\frac{1}{2}(n-1)$, whichever is an integer.

Using this result and afterwards substituting from (3), we have

$$\begin{aligned} & r^n P_n^m(\cos\theta)\cos m\phi \\ &= \frac{i^{m-n}}{2\pi}\,\frac{(n+m)!}{n!}\int_0^{2\pi} \xi^n \cos mt\,dt \\ &= \frac{c^n}{2\pi i^{n-m}}(n+m)!\int_0^{2\pi}\sum_{p=0}^{q}\frac{(2n-4p+1)(n-p)!}{(2n-2p+1)!\,p!}\,2^{n-2p}\,P_{n-2p}\left(\frac{\xi}{c}\right)\cos mt\,dt \\ &= \frac{c^n}{i^{n-m}}(n+m)!\cos m\phi \\ &\qquad \times \sum_{p=0}^{q}\frac{(2n-4p+1)(n-p)!\,(n-2p-m)!}{(2n-2p+1)!\,(n-2p+m)!\,p!}\,2^{n-2p}P_{n-2p}^m(i\xi)\,P_{n-2p}^m(\cos\eta). \quad (6) \end{aligned}$$

If n is a negative integer, let it be $-\nu-1$; then, m being a positive integer,

$$\frac{\Gamma(n+m+1)}{\Gamma(n+1)} = \frac{\sin(\nu+1)\pi}{\sin(\nu-m+1)\pi}\,\frac{\Gamma(\nu+1)}{\Gamma(\nu-m+1)} = (-1)^m\frac{\nu!}{(\nu-m)!},$$

and (1) may be written

$$\frac{1}{r^{\nu+1}}P_\nu^m(\cos\theta)\cos m\phi = \frac{i^{\nu-m+1}}{2\pi}\,\frac{\nu!}{(\nu-m)!}\int_0^{2\pi}\xi^{-\nu-1}\cos mt\,dt. \qquad (7)$$

Now $\xi^{-\nu-1}$ regarded as a function of the complex variable ξ is holomorphic at all points outside any ellipse C whose foci are the points ± 1. It may therefore be expanded by Neumann's theorem in a series of Legendre functions of the second kind. It may be shown that if ξ lies outside an ellipse whose foci are $\pm c$,

$$\xi^{-\nu-1} = \frac{1}{c^{\nu+1}\nu!}\sum_{p=0}^{\infty}(-1)^p\frac{(2\nu+4p+1)(2\nu+2p)!}{2^{\nu+2p}(\nu+p)!\,p!}Q_{\nu+2p}\left(\frac{\xi}{c}\right).$$

The spherical harmonics which vanish at infinity are therefore given by

$$\frac{1}{r^{\nu+1}}P_\nu^m(\cos\theta)\cos m\phi$$

$$= \frac{i^{\nu-m+1}}{2\pi c^{\nu+1}}\,\frac{1}{(\nu-m)!}\int_0^{2\pi}\sum_{p=0}^{\infty}(-1)^p\frac{(2\nu+4p+1)(2\nu+2p)!}{2^{\nu+2p}(\nu+p)!\,p!}Q_{\nu+2p}\left(\frac{\xi}{c}\right)\cos mt\,dt$$

$$= \frac{i^{\nu-m+1}}{c(\nu-m)!}\cos m\phi$$

$$\times\sum_{p=0}^{\infty}(-1)^p\,\frac{(2\nu+4p+1)(2\nu+2p)!\,(\nu+2p-m)!}{2^{\nu+2p}(\nu+p)!\,(\nu+2p+m)!\,p!}Q_{\nu+2p}^m(i\xi)P_{\nu+2p}^m(\cos\eta), \quad (8)$$

by equation (5).

In arriving at this result we have assumed that ξ lies outside any ellipse of a certain family. This is equivalent to the assumption that ξ is not a real quantity between $\pm|c|$. It is readily seen that this condition is satisfied if $z>|c|$, whether c be real or imaginary. Now the series on the right-hand side of (8) converges uniformly if $i\xi$ is finitely different from unity, and allowing for the fact that ξ, η are not at all points continuous functions of ϖ, z, its space gradient is continuous if $\xi\neq 0$. In the case of both oblate and problate spheroids these conditions are satisfied outside any spheroid of the system $\xi=$ const. In this region both sides of (8) are solutions of Laplace's equation, which, with their first differential coefficients, are finite and continuous. They have been proved equal when $z>|c|$; hence (8) is true everywhere outside any ellipse $\xi=$ const.

To transform spherical to cylindrical harmonics, having the same axis and origin.—From Hankel's contour integral for the Γ-function we readily obtain

$$\Gamma(n+1)\zeta^{-n-1} = i^{-n-1}\int_0^\infty e^{ik\zeta}k^n dk, \tag{9}$$

provided that $n+1>0$ and the real part of $i\zeta$ is negative, *i.e.*, $z>0$.

Hence

$$\begin{aligned}\frac{1}{r^{n+1}}P_n^m(\cos\theta)\cos m\phi &= \frac{i^{n-m+1}}{2\pi}\frac{n!}{(n-m)!}\int_0^{2\pi}\zeta^{-n-1}\cos mt\,dt\\ &= \frac{i^{-m}}{2\pi(n-m)!}\int_0^{2\pi}\int_0^\infty e^{ik\zeta}k^n\cos mt\,dk\,dt\\ &= \frac{\cos m\phi}{(n-m)!}\int_0^\infty e^{-kz}J_m(k\varpi)\,k^n dk, \end{aligned}\tag{9}$$

by (2), if $z>0$.

This result was given by Hobson.* It expresses any spherical harmonic which vanishes at infinity in terms of Bessel functions. Hobson also gave a contour integral, involving Bessel functions of complex argument, for the spherical harmonics which are finite at the origin. This may be obtained in the same way.

To transform cylindrical to spherical harmonics. — Expanding the exponential in (2) in powers of its argument and then using (1), we have

$$\begin{aligned}e^{-kz}J_m(k\varpi)\cos m\phi &= \frac{i^{-m}}{2\pi}\int_0^{2\pi}e^{ik\zeta}\cos mt\,dt\\ &= \frac{i^{-m}}{2\pi}\int_0^{2\pi}\sum_{p=0}^{\infty}\frac{(ik)^p}{p!}\zeta^p\cos mt\,dt\\ &= \cos m\phi\sum_{p=0}^{\infty}\frac{(-1)^{m+p}}{(m+p)!}k^p r^p P_p^m(\cos\theta),\end{aligned}$$

or, changing the sign of k,

$$e^{kz}J_m(k\varpi)\cos m\phi = \cos m\phi\sum_{p=0}^{\infty}\frac{(kr)^p}{(m+p)!}P_p^m(\cos\theta). \tag{11}$$

The result was also given by Hobson in the paper referred to above. It has been proved for $z>|c|$, but, both sides of the equation (11) being

* "Bessel's Functions and Relations connecting them with Spherical Harmonics," *Proc. London Math. Soc.*, 1893, Vol. xxv, p. 73.

continuous potential functions everywhere in the finite region, this restriction may be removed.

To transform cylindrical to spheroidal harmonics, having the same origin and axes.—Bauer* has given the expansion

$$e^{ik\zeta} = \left(\frac{2\pi}{ck}\right)^{\frac{1}{2}} \sum_{n=0}^{\infty} (2n+1)\, i^n J_{n+\frac{1}{2}}(ck)\, P_n\left(\frac{\xi}{c}\right).$$

By Neumann's theorem on the expansion of a function in a series of Legendre functions, the expansion is valid for all finite values of ξ, for $e^{ik\zeta}$ is holomorphic for all finite values of ξ. From this result, together with (2) and (3), we have

$$e^{-ikz} J_m(k\varpi)\cos m\phi$$

$$= \left(\frac{2\pi}{ck}\right)^{\frac{1}{2}} \cos m\phi \sum_{n=0}^{\infty} i^{n-m}(2n+1)\,\frac{(n-m)!}{(n+m)!}\, J_{n+\frac{1}{2}}(ck)\, P_n^m(i\xi)\, P_n^m(\cos\eta). \quad (12)$$

This result is valid for all finite values of ξ.

To transform spheroidal to spherical harmonics, having the same axes and origin.—First consider the internal spheroidal harmonics. We have

$$P_n^m(i\xi)\, P_n^m(\cos\eta)\cos m\phi = \frac{1}{2\pi}\,\frac{(n+m)!}{(n-m)!}\int_0^{2\pi} P_n\left(\frac{\xi}{c}\right)\cos mt\, dt.$$

Now $$P_n\left(\frac{\xi}{c}\right) = \sum_{p=0}^{q} (-1)^p \frac{(2n-p)!}{2^n p!\,(n-p)!\,(n-2p)!}\left(\frac{\xi}{c}\right)^{n-2p},$$

where $q = \frac{1}{2}n$ or $\frac{1}{2}(n-1)$, whichever is an integer. The corresponding transformations when n is not an integer, but the value of ξ is suitably restricted, can be obtained by the same method if they are required. From the above results and (1) we have

$$P_n^m(i\xi)\, P_n^m(\cos\eta)\cos m\phi$$

$$= \frac{i^{n-m}}{2^n}\,\frac{(n+m)!}{(n-m)!}\cos m\phi \sum_{p=0}^{q} \frac{(2n-p)!}{(n-p)!\,(n+m-2p)!\,p!}\left(\frac{r}{c}\right)^{n-2p} P_{n-2p}^m(\cos\theta). \quad (13)$$

The external harmonics are given by

$$Q_n^m(i\xi)\, P_n^m(\cos\eta)\cos m\phi = \frac{1}{2\pi}\,\frac{(n+m)!}{(n-m)!}\int_0^{2\pi} Q_n\left(\frac{\xi}{c}\right)\cos mt\, dt,$$

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* *Crelle's Journal*, LVI, p. 104.

and $$Q_n\left(\frac{\xi}{c}\right) = 2^n \sum_{p=0}^{\infty} \frac{(n+p)!\,(n+2p)!}{(2n+2p+1)!\,p!} \left(\frac{c}{\xi}\right)^{n+2p+1},$$

provided that $$|\, x\cos t + y\sin t + iz \,| > |\, c \,|$$

for all real values of t, *i.e.* that

$$z > |\, c \,|.$$

Hence, using (7),

$$Q_n^m(i\xi)\, P_n^m(\cos\eta)\cos m\phi$$

$$= \frac{2^{n-1}}{\pi} \frac{(n+m)!}{(n-m)!} \int_0^{2\pi} \sum_{p=0}^{\infty} \frac{(n+p)!\,(n+2p)!}{(2n+2p+1)!\,p!} \left(\frac{c}{\xi}\right)^{n+2p+1} \cos mt\, dt$$

$$= \frac{2^n}{i^{n-m+1}} \frac{(n+m)!}{(n-m)!} \cos m\phi \sum_{p=0}^{\infty} \frac{(n+p)!\,(n-m+2p)!}{(2n+2p+1)!\,p!} \left(\frac{c}{r}\right)^{n+2p+1} P_{n+2p}^m(\cos\theta). \tag{14}$$

By the same argument as we have used above, (13) is true everywhere, and (14) is true outside the sphere $r = |\, c \,|$.

To transform spheroidal to cylindrical harmonics.—If $z > |\, c \,|$, we have

$$Q_n\left(\frac{\xi}{c}\right) = 2^n \sum_{p=0}^{\infty} \frac{(n+p)!\,(n+2p)!}{(2n+2p+1)!\,p!} \left(\frac{c}{\xi}\right)^{n+2p+1},$$

and from (9) in the same circumstances

$$\frac{1}{\xi^{n+2p+1}} = \frac{(-1)^p}{i^{n+1}(n+2p)!} \int_0^{\infty} e^{ik\zeta} k^{n+2p}\, dk.$$

Hence $$Q_n\left(\frac{\xi}{c}\right) = \frac{2^n c}{i^{n+1}} \sum_{p=0}^{\infty} \frac{(-1)^p\,(n+p)!}{(2n+2p+1)!\,p!} \int_0^{\infty} e^{ik\zeta} (ck)^{n+2p}\, dk.$$

We may interchange the order of the summation and integration and, after reducing the factorials, we have

$$Q_n\left(\frac{\xi}{c}\right) = \frac{c\Gamma(\frac{1}{2})}{2i^{n+1}} \int_0^{\infty} e^{ik\zeta} \sum_{p=0}^{\infty} \frac{(-1)^p}{p!\,\Gamma(n+p+\frac{3}{2})} \left(\frac{ck}{2}\right)^{n+2p} dk,$$

which, by the series for the Bessel functions, gives

$$Q_n\left(\frac{\xi}{c}\right) = i^{-n-1} \int_0^{\infty} \left(\frac{c\pi}{2k}\right)^{\frac{1}{2}} e^{ik\zeta} J_{n+\frac{1}{2}}(ck)\, dk.$$

Hence, using (2) and (5), we have without further difficulty

$$Q_n^m(i\xi)\,P_n^m(\cos\eta)\cos m\phi$$

$$= i^{m-n-1}\frac{(n+m)!}{(n-m)!}\cos m\phi\int_0^\infty\left(\frac{c\pi}{2k}\right)^{\frac{1}{2}}J_{n+\frac{1}{2}}(ck)\,e^{-kz}J_m(k\varpi)\,dk, \quad (15)$$

if $z>0$.

Whittaker's integral may also be used to obtain relations between harmonics of the same kind referred to different axes. No great generality can be expected in this direction, as the integral itself does not admit some very simple changes of axes. In some cases, however, the formulæ of transformation are very readily obtained. Amongst those we may mention Bessel's formulæ for changing the origin of spherical harmonics whilst retaining the same axis, and Nicholson's formulæ connecting Bessel harmonics referred to parallel axes.

In this paper we have confined our attention to three important types of harmonics, but the same method may be used to investigate the relations between these and ellipsoidal, paraboloidal, toroidal, bipolar, and elliptic and parabolic cylinder harmonics, all of which may be expressed in Whittaker's form.

THEOREMS ON CONFORMAL TRANSFORMATION

By J. G. Leathem.

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1. A conformal curve-factor in a complex variable $\zeta\ (\equiv \xi+i\eta)$ has been defined* as a function $\mathscr{C}(\zeta)$, such that the transformation

$$dz = \mathscr{C}(\zeta)\, d\zeta$$

specifies a conformal correspondence between the half-plane $\eta > 0$ and some definite region in the plane of z. Particular classes of curve-factor are :—(i) pure curve-factors, for which the boundary of the relevant region in the z plane, whether wholly or only partially curved, is free from corners ; (ii) corner-factors or Schwarzian factors, of the kind familiar in the Schwarz-Christoffel transformation.

Use has already been made of the fairly obvious fact that the limit of a product of corner-factors with infinitesimal exponents is itself a curve-factor. The present short piece of work was undertaken with a view to proving a converse theorem, namely that every curve-factor is expressible as a limit product of corner-factors, that is as the exponential of a certain line-integral taken round the boundary. This result is established, and is shewn to have general theoretical consequences, including the theorem that any curve-factor is expressible as a product of corner-factors and pure simple† curve-factors. Corresponding results are obtained for periodic curve-factors.

A second and quite different expression for a curve-factor as the exponential of a line-integral round the boundary of the relevant region results from the same analysis, and is of some interest.

2. A general conformal curve-factor $\mathscr{C}(\zeta)$, as defined above, has no zeros, infinities or branch-points in the principal half-plane of ζ, *i.e.*, the half-plane $\eta > 0$. It may have branch-points on the real axis, as these do not complicate operations confined to the principal half-plane. It may

* J. G. Leathem, "Some Applications of Conformal Transformation to Problems in Hydrodynamics," *R. S. Phil. Trans.*, Sec. A, Vol. 215, 1915.

† *L.c.*, § 4.

have zeros or infinities at points on the real axis, but these will be restricted to the type in which the limit form is $A(\xi-a)^p$, where $1 \geqslant p > -1$.

It will be assumed that $\mathscr{C}(\xi)$ has a definite angular range $n\pi$, and therefore a definite order at infinity n, so that the form of the function for $|\xi|$ great is

$$\mathscr{C}(\xi) = K\xi^n\{1+f(\xi)\},$$

where n is real and $f(\xi) \to 0$ for $|\xi| \to \infty$. There is no loss of generality in assuming K real. The vanishing of $f(\xi)$ must be such that

$$\xi f'(\xi) \log \xi \to 0,$$

as is true, for example, when $f(\xi)$ is a sum of negative powers of ξ.

3. Let ξ_0 be a value of ξ corresponding to a point in the principal half-plane. It is proposed, in the first instance, to consider the integral

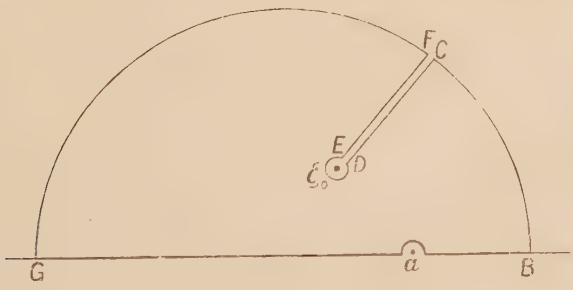
$$\int (1/\pi) \log (\xi_0-\xi)\, d \log \mathscr{C}(\xi),$$

taken round a contour in the principal half-plane of ξ, consisting mainly of the real axis and a semicircle of great radius R with its centre at the origin. The formulæ will be pressed to limit forms corresponding to R infinite.

Log $\mathscr{C}(\xi)$ has neither infinity nor branch-point in the region enclosed by this contour. But ξ_0 is in the region and is a branch-point of $\log(\xi_0-\xi)$; accordingly ξ_0 is surrounded by a circular cavity of infinitesimal radius ϵ, and a cut is made from this cavity to some point on the semicircle. The infinitesimal circumference and the two sides of the cut are added to the contour.

An infinity or zero of $\mathscr{C}(\xi)$ on the real axis, typified by $\xi = a$, is an infinity of $\log \mathscr{C}(\xi)$. Near such a point the contour must be modified, being kept clear of the point of infinity by a semicircular detour of infinitesimal radius ϵ' on the positive side of the axis.

With these modifications the contour assumes the form typified in the



diagram, and encloses a region in which the subject of integration has no infinities or branch points. The contour integral must therefore equal zero.

The logarithm may be so chosen that, as ξ traverses the real axis from left to right, the vector-angle of $\log(\xi_0-\xi)$ increases from zero to π.

As ξ makes a circuit round ξ_0 in the positive sense, $\log(\xi_0-\xi)$ increases by $2i\pi$. Hence, by taking together corresponding elements of integration on opposite sides of the cut, it is seen that

$$\left(\int_C^D+\int_E^F\right)\frac{1}{\pi}\log(\xi_0-\xi)\,d\log\mathscr{C}(\xi)$$
$$=2i\int_C^D d\log\mathscr{C}(\xi)\to 2i\{\log\mathscr{C}(\xi_0)-\log\mathscr{C}(\xi_C)\}.$$

On the infinitesimal circumference round ξ_0, $d\xi$ is proportional to ϵ, and $\epsilon\log\epsilon\to 0$, so this circumference makes no contribution to the contour integral, in the limit for $\epsilon\to 0$.

As ξ traverses the great semicircle there is a sudden drop of $2i\pi$ in the value of $\log(\xi_0-\xi)$ in passing from C to F. Continuity is restored, for the purpose of this part of the integration, by continuing the value of $\log(\xi_0-\xi)$, as from C, across the gap; this necessitates adding

$$-2i\int_F^G d\log\mathscr{C}(\xi)=-2i\{\log\mathscr{C}(\xi_G)-\log\mathscr{C}(\xi_C)\}.$$

On the great semicircle, approximately,

$$d\log\mathscr{C}(\xi)=[n\xi^{-1}+f'(\xi)\{1+f(\xi)\}^{-1}]\,d\xi,$$

and, if $\xi=R\exp i\theta$,

$$\log(\xi_0-\xi)=\log R+i(\pi+\theta)+\text{negative powers of } R\exp i\theta.$$

So the integral on the semicircle has the same limit as

$$\frac{i}{\pi}\int_0^\pi\{n+\xi f'(\xi)\}\{\log R+i(\pi+\theta)\}\,d\theta,$$

or

$$in\log R-(\tfrac{3}{2})n\pi.$$

On the infinitesimal semicircle round a it suffices to put $\log(\xi_0-a)$ instead of $\log(\xi_0-\xi)$, and $p(\xi-a)^{-1}\,d\xi$ instead of $d\log\mathscr{C}(\xi)$. If

$$\xi-a=\epsilon'\exp i\theta',$$

the integration with respect to θ' is from π to zero, and ϵ' disappears. So the limit of the integral is

$$-ip\log(\xi_0-a).$$

The line-integral along the real axis is generally semi-convergent as regards the infinity at a, and the choosing of an infinitesimal gap with a in the middle gives the Cauchy principal value. This is indicated by the letters $\boldsymbol{pv}$.

On collection of the various parts of the contour-integral, and equation of the sum to zero, there appears the following formula, which must be understood as only approximate for any definite large value of R, but absolutely true in the limit for $R \to \infty$:—

$$2i \log \mathscr{C}(\xi_0) - 2i \log \mathscr{C}(\xi_G) + in \log R - (\tfrac{3}{2})n\pi$$

$$-i\Sigma p \log (\xi_0 - a) + \boldsymbol{pv} \int_G^B \frac{1}{\pi} \log (\xi_0 - \xi)\, d \log \mathscr{C}(\xi) = 0. \quad (1)$$

It is convenient to alter the form of the last integral, wherein ξ is real, by putting

$$\mathscr{C}(\xi) = q \exp i\chi, \quad (2)$$

so that

$$d \log \mathscr{C}(\xi) = d \log q + i d\chi.$$

The infinity of $\mathscr{C}(\xi)$, for $\xi = a$, has to do with q and not with χ; hence the part of the integral depending on χ is absolutely convergent, and does not require the symbol $\boldsymbol{pv}$. The result now takes the form :—

$$2i \{\log \mathscr{C}(\xi_0) - \log \mathscr{C}(\xi_G)\} + in \log R - (\tfrac{3}{2})n\pi$$

$$-i\Sigma p \log (\xi_0 - a) + \boldsymbol{pv} \int_G^B \frac{1}{\pi} \log (\xi_0 - \xi)\, d \log q + i \int_G^B \frac{1}{\pi} \log (\xi_0 - \xi)\, d\chi = 0. \quad (3)$$

4. Let ξ_0' be the complex conjugate to ξ_0, so that the two quantities are represented by points which are images of one another with respect to the real axis. And let the integral

$$\int (1/\pi) \log (\xi_0' - \xi)\, d \log \mathscr{C}(\xi)$$

be taken round a contour identical with that of the previous article in all respects save one. As ξ_0' is not situated in the region enclosed by the contour, the cut and the infinitesimal cavity which would be necessary for the treatment of a branch-point are not now needed. Thus the contour consists of the great and infinitesimal semicircles and the real axis, and the contour integral must be zero.

Log $(\xi_0' - \xi)$ will be so defined that its vector-angle decreases from zero to $-\pi$ as ξ traverses the real axis from left to right.

The contributions to the contour integral arising from the cut and the

discontinuity at its outer extremity are not represented in the present evaluation, but the other parts of the integral correspond closely to those of the previous article. So the equality which now takes the place of formula (1) is readily shewn to be

$$in \log R + \tfrac{1}{2}n\pi - i\Sigma p \log(\xi_0' - a) + \boldsymbol{pv} \int_G^B \frac{1}{\pi} \log(\xi_0' - \xi)\, d \log \mathscr{C}(\xi) = 0, \quad (4)$$

which is the same as

$$in \log R + \tfrac{1}{2}n\pi - i\Sigma p \log(\xi_0' - a) + \boldsymbol{pv} \int_G^B \frac{1}{\pi} \log(\xi_0' - \xi)\, d \log q + i \int_G^B \frac{1}{\pi} \log(\xi_0' - \xi)\, d\chi = 0. \quad (5)$$

In this formula let the sign of i be changed wherever it occurs, which involves also changing ξ_0' to ξ_0. The result is

$$-in \log R + \tfrac{1}{2}n\pi + i\Sigma p \log(\xi_0 - a) + \boldsymbol{pv} \int_G^B \frac{1}{\pi} \log(\xi_0 - \xi)\, d \log q - i \int_G^B \frac{1}{\pi} \log(\xi_0 - \xi)\, d\chi = 0. \quad (6)$$

5. From formulæ (3) and (6) either of the two line-integrals can be eliminated by addition or subtraction, as the case may be. After slight rearrangement the results take the forms

$$\log \mathscr{C}(\xi_0) = \log \mathscr{C}(\xi_G) - \tfrac{1}{2}in\pi + i\,\boldsymbol{pv} \int_G^B \frac{1}{\pi} \log(\xi_0 - \xi)\, d \log q, \quad (7)$$

$$\log \mathscr{C}(\xi_0) = \log \mathscr{C}(\xi_G) - in\pi - n \log R + \Sigma p \log(\xi_0 - a) - \int_G^B \frac{1}{\pi} \log(\xi_0 - \xi)\, d\chi. \quad (8)$$

When R becomes great without limit

$$\log \mathscr{C}(\xi_G) - n \log R \to \log K + ni\pi,$$

and so the limit forms of formulæ (7) and (8) are

$$\log \mathscr{C}(\xi_0) = \operatorname*{Lim}_{R \to \infty} \left\{ \log K + n \log R + \tfrac{1}{2}ni\pi + i\,\boldsymbol{pv} \int_{\xi=-R}^{\xi=R} \frac{1}{\pi} \log(\xi_0 - \xi)\, d \log q \right\}, \quad (9)$$

$$\log \mathscr{C}(\xi_0) = \log K + \Sigma p \log(\xi_0 - a) - \int_{\xi=-\infty}^{\xi=\infty} \frac{1}{\pi} \log(\xi_0 - \xi)\, d\chi. \quad (10)$$

These are two distinct expressions for the logarithm of a curve-factor as line-integrals along the axis of ξ real; they imply corresponding expressions for the curve-factor as exponentials of such line-integrals.

6. Of formulæ (9) and (10) the latter is probably the more useful and interesting. Its significance depends upon the geometrical interpretation of χ in connexion with the relation $dz = \mathscr{C}(\xi)d\xi$. When ξ is real, and so replaceable by ξ, formula (2) shews that

$$dz = q \exp(i\chi) d\xi,$$

wherein dz represents vectorially the element of arc (say ds) of the boundary of that region in the z plane which is conformally represented on the principal half-plane of ξ. Hence χ is the angle which the tangent to the boundary makes with the real axis in the z plane.

Formula (10) is equivalent to

$$\mathscr{C}(\xi_0) = K\Pi(\xi_0 - a)^p \operatorname{Lim} \Pi(\xi_0 - \xi)^{-d\chi/\pi}, \qquad (11)$$

and exhibits the general curve-factor as the limit of a product of corner-factors, these having definite exponents for all corners in the boundary, and infinitesimal exponents for all points where the boundary is smoothly curved. The terms of the product correspond to values of ξ ranging over the whole boundary.

This confirms a guess hazarded in the previous paper already cited, to the effect that the most general conformal transformation for the half-plane of ξ is of the type of formula (11), or, in other words, that every curve-factor is equal to the limit of a product of corner-factors.

7. A curve-factor of restricted linear range is one for which the boundary in the z plane consists partly of one or more definite curved portions, and, for the rest, of straight lines with which the curves join smoothly. For such a curve-factor $d\chi$ differs from zero only along one or more definite stretches of the real axis of ξ, which constitute the linear range of curvilinearity, and it is only values of ξ included in these stretches that contribute to the integral of formula (10) or the product of formula (11).

Now since the product corresponding to a linear range made up of two separate portions is itself the product of two products, one corresponding to each of the sub-ranges, it follows that any curve-factor of multiple linear range can be factorised into curve-factors each having a simple linear range. This property of multiple curve-factors was surmised, but not proved, in the writer's previous paper (§ 29) referred to above.

8. If the functional relation between χ and ξ be $\chi = f(\xi)$, and if there be no corners, formula (10) is equivalent to

$$\mathscr{C}(\xi_0) = K \exp\{-(1/\pi)\int f'(\xi)\log(\xi_0-\xi)\,d\xi\}, \qquad (12)$$

and in this form it constitutes the most general formula for a curve-factor of assigned linear and angular range. If the linear range is from $\xi = b$ to $\xi = c$, these must be taken as the limits of the line-integral; and, if the angular range is required to be $\kappa\pi$, f must be so chosen that

$$f(b)-f(c) = \kappa.$$

Any form may be assigned to f which does not introduce singularities already expressly excluded, and there is a form of f to correspond to every possible form of $\mathscr{C}$.

The formula may be used for the calculation of forms of curve-factor, and the only limitation upon this method of generating fresh types is the possible difficulty of effecting the integration. The indefinite integral is readily found if $f(\xi)$ is a rational algebraical function of ξ, or if it is the inverse of the square root of a linear or quadratic function. For example, the forms of $\mathscr{C}$ corresponding to

$$f(\xi) = 1/(\xi-k) \quad \text{or} \quad f(\xi) = 1/\sqrt{(\xi-k)}$$

can be calculated with little trouble. For some forms of f the definite integral could probably be calculated otherwise than through the indefinite integral. Thus the formula is substantially useful for extending the category of available curve-factors. Nevertheless its practical limitations leave plenty of scope for such other modes of generating fresh types as may result from different lines of investigation.

It is perhaps worth noting that

$$f'(\xi) = d\chi/d\xi = (d\chi/ds)(ds/d\xi) = q/\rho,$$

where ρ is the radius of curvature of the boundary in the z plane, and q the linear magnification in the z plane as compared with the ξ plane at the point considered.

9. Formula (9) is equally available, but probably less useful, for the calculation of forms of curve-factor; it has not the geometrically advantageous feature possessed by formula (10), namely that only the curved portions of the boundary contribute to the integral in the latter.

The explicit appearance of $\log R$ in formula (9) gives an appearance of divergence which is fallacious; the term actually serves to cancel a diver-

gence of the definite integral, and the limit on the right-hand side of the formula is definite. Peculiarities arising out of corners in the boundary, instead of appearing explicitly as in formula (10), manifest themselves in the semiconvergences which necessitate the restriction of the integral to the Cauchy principal value.

Formula (9) is susceptible of an interpretation of interest in hydrodynamics. If ξ be taken to be the velocity-potential and η the stream-function of two-dimensional steady continuous liquid flow, the relation $dz = \mathscr{C}(\zeta)\,d\zeta$ specifies a configuration of flow in the z plane in which the boundary is the curve corresponding to $\eta = 0$. In this boundary the velocity is tangential, and its magnitude is $|\,d\zeta/dz\,|$ or $1/q$. Now, if part of the boundary be a free stream-line, in the absence of a field of force the velocity is constant in magnitude along this line, and so $d\log q = 0$. Thus such parts of the boundary as are free stream-lines make no contribution to the integral in formula (9). The equivalent formula,

$$\mathscr{C}(\zeta_0) = Ke^{\frac{1}{2}ni\pi} \operatorname*{Lim}_{R\to\infty} \left\{ R^n \exp i\,\boldsymbol{pv} \int \frac{1}{\pi} \log(\zeta_0 - \xi)\, d\log q \right\}, \qquad (13)$$

constitutes the most general specification of a motion with free stream-lines, and is in terms of an integral whose range of integration includes only those parts of the boundary which are fixed. By suitable choice of functional relations between q and ξ, with introduction of appropriate singularities for the corners, it is theoretically possible to make this formula represent continuous motion with free stream-lines past all types of fixed boundary.

If it were desired to use the formula for generating new specifications of motion with free stream-lines, it would seem best to take separate account of corners in the boundary. Thus, in the case of a free stream-line corresponding to the linear range $-\infty < \xi < 0$, it is known* that an angle $(1-p)\,\pi$ in the fixed boundary at $\xi = a$ is represented in the curve-factor

$$(a^{\frac{1}{2}} + \zeta^{\frac{1}{2}})^{2p}/(\zeta - a)^p, \qquad (14)$$

whose modulus is constant for ζ real and negative. Hence, if P be the product of such functions representing all the corners, the transformation $dz = P\,\mathscr{C}_s(\zeta)\,d\zeta$, where

$$\mathscr{C}_s(\zeta) = Ke^{\frac{1}{2}ni\pi} \operatorname*{Lim}_{R\to\infty} \left\{ R^n \exp i \int_0^R \frac{1}{\pi} \log(\zeta - \xi)\, df(\xi) \right\}, \qquad (15)$$

and $f(\xi)$ has no infinities, gives a motion of the type required.

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* *L.c.*, § 36.

For example, the assumption

$$f(\xi) = \pi\xi/(c+\xi),$$

where $c > 0$, leads through formula (15) to the curve-factor

$$\zeta^{i\zeta/(c+\zeta)}(-c)^{ic/(c+\zeta)},$$

which has constant modulus for ζ real and negative. This could be combined with factors of the type of formula (14).

10. *Periodic Conformal Curve-Factors.* — A periodic curve-factor of linear period λ is a function $\mathscr{C}(\zeta)$ such that the transformation $dz = \mathscr{C}(\zeta)\,d\zeta$ represents a doubly connected region in the z plane, having an internal but no external boundary, conformally and repeatedly upon successive semi-infinite strips of width λ in the principal half-plane of ζ. The writer has given an account of these functions elsewhere.* The typical periodic corner-factor is a power of $\sin\{\pi\lambda^{-1}(\zeta-a)\}$.

The proof for periodic curve-factors of the theorems analogous to those of formulæ (9) and (10) depends on the application of two contour integrations to almost the same contour in the ζ plane. The integral is, in the one case

$$\int \frac{1}{\pi} \log \sin \frac{\pi}{\lambda}(\xi_0-\zeta)\,d \log \mathscr{C}(\zeta),$$

and, in the other, the similar expression with the conjugate complex ξ_0' substituted for ξ_0. The contour consists, in the main, of the rectangle determined by $\xi = a$, $\xi = a+\lambda$, $\eta = 0$, $\eta = t$, where $t > 0$; but, as ξ_0 is chosen to be within this rectangle, the first integration must include an infinitesimal cavity round ξ_0 and the two sides of a straight cut drawn from ξ_0 perpendicularly to the side $\eta = t$.

$\mathscr{C}(\zeta)$ is supposed to have a definite angular period $2n\pi$, and therefore a definite exponential order at infinity n, so that, for η great and positive,

$$\mathscr{C}(\zeta) = \exp(-2\pi ni\zeta/\lambda)\,\Sigma\, c_s \exp(2\pi si\zeta/\lambda), \quad (s = 0, 1, 2, \ldots, \infty).$$

It is contemplated that the formulæ obtained will be pressed to limit forms corresponding to t infinite.

As details of a contour integration having considerable resemblance to those now required are included in the paper just referred to (§ 11), the corresponding work may be omitted here, and the results stated without

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* "On Periodic Conformal Curve-Factors and Corner-Factors," *Proc. Royal Irish Academy*, Vol. XXXIII, Sec. A, 1916.

proof. The first integration gives

$$\int_a^{a+\lambda} \frac{1}{\pi} \log \sin \left\{ \frac{\pi}{\lambda} (\xi_0 - \xi) \right\} d \log \mathscr{C}(\xi) + 2n \{ \pi - i \log 2 - \pi \xi_0 / \lambda \} + i \{ 2 \log \mathscr{C}(\xi_0) - \log \mathscr{C}(a) - \log c_0 \} = 0, \quad (16)$$

and the second gives

$$\int_a^{a+\lambda} \frac{1}{\pi} \log \sin \left\{ \frac{\pi}{\lambda} (\xi_0' - \xi) \right\} d \log \mathscr{C}(\xi) + 2n \{ \pi - i \log 2 - \pi \xi_0' / \lambda \} + i \{ \log \mathscr{C}(a) - \log c_0 \} = 0. \quad (17)$$

As before, let $\mathscr{C}(\xi) = q \exp(i\chi)$; let the sign of i be changed wherever it occurs in equation (17), which involves also changing ξ_0' to ξ_0; and let the resulting equation be combined with (16) by addition and subtraction. There emerge the formulæ :—

$$\log \mathscr{C}(\xi_0) = i \int_a^{a+\lambda} \frac{1}{\pi} \log \sin \left\{ \frac{\pi}{\lambda} (\xi_0 - \xi) \right\} d \log q + \log \mathscr{C}(a) + 2ni\pi \left(1 - \frac{\xi_0}{\lambda} \right), \quad (18)$$

$$\log \mathscr{C}(\xi_0) = - \int_a^{a+\lambda} \frac{1}{\pi} \log \sin \left\{ \frac{\pi}{\lambda} (\xi_0 - \xi) \right\} d\chi + 2n \log 2 + \log c_0. \quad (19)$$

These are analogous to formulæ (9) and (10). Formula (19) is equivalent to

$$\mathscr{C}(\xi_0) = 2^{2n} c_0 \exp \left[- \int_0^{2n\pi} \frac{1}{\pi} \log \sin \left\{ \frac{\pi}{\lambda} (\xi_0 - \xi) \right\} d\chi \right], \quad (20)$$

and exhibits the general periodic curve-factor as the limit of a product of periodic corner factors. The inferences which can be drawn from this result correspond to those made in articles 6 to 8 above.

BESSEL FUNCTIONS AND KAPTEYN SERIES

By G. N. WATSON.

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1. A series of the type

$$a_0 J_\nu(\nu z)+a_1 J_{\nu+1}\{(\nu+1)z\}+a_2 J_{\nu+2}\{(\nu+2)z\}+\ldots+a_n J_{\nu+n}\{(\nu+n)z\}+\ldots,$$

in which the $(n+1)$-th term is a constant multiple of the Bessel function of the first kind with argument $(\nu+n)z$ and of order $\nu+n$, is usually called a Kapteyn series; to avoid unnecessary complications produced by an infinity at the origin, it is customary to assume that $\nu \geqslant 0$, although, on the other hand, the further restriction that ν should be an integer is *not* imposed on the series.

Series of this type occur in the Laplace-Bessel solution* of Kepler's astronomical problem of expressing the eccentric anomaly in terms of the mean anomaly; and related series appear in many problems of a similar nature. The first writer to investigate such series in a systematic manner was Kapteyn,† while more recent researches are due to Nielsen.‡

The related topic of the behaviour of $J_n(nz)$ when n is large and positive, but not necessarily an integer, while z is fixed, has been discussed in two important papers by Debye,§ who has obtained complete asymptotic expansions when n is large. Some isolated results concerning the dominant terms in these expansions had, however, been discovered

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* Laplace, *Mécanique Céleste*, v, *Supplément;* Bessel, *Berliner Abhandlungen*, 1824. It should be noted that Laplace obtains rigorously an asymptotic formula for $I_n(nx)$ when n is large; he then states that the corresponding formula for $J_n(nx)$ is true (which cannot justifiably be inferred from the previous result) and that he has an independent proof of it, which, however, he does not give.

† *Ann. de l'Ecole Normale Supérieure*, (3) x, 1893, pp. 91–120.

‡ Nielsen, *Handbuch der Theorie der Cylinderfunktionen*, 1904, gives an account of his researches with references to memoirs by Kapteyn and by himself.

§ *Mathematische Annalen*, LXVII, 1909, pp. 535–558; *Münchener Sitzungsberichte* (5), 1910.

previously by Laplace, by Carlini,* by Graf and Gubler,† and by Nicholson.‡

The investigations by Kapteyn have been mainly concerned with the problem of expanding an arbitrary analytic function into a Kapteyn series, and of this problem he has given a complete solution; for example, one of his results is that, if $f(z)$ is analytic throughout the interior of the circle $|z| = 1$, then $f(z)$ is expansible into the Kapteyn series

$$a_0 + \sum_{n=1}^{\infty} a_n J_n(nz),$$

the expansion being valid throughout the interior of a certain curve C in the z-plane; and expressions are given for the coefficients a_n in the form of contour integrals. The polar coordinates (r, θ) of any point on the curve C are given in terms of a parameter by the equations

$$r = \sqrt{\frac{2t}{\sinh 2t}}, \qquad \cos 2\theta = \cosh 2t - t \sinh 2t.$$

Nielsen's investigations are more particularly concerned with the expansion of z^ν into a Kapteyn series. It may be remarked that his mode of procedure is slightly crude, as his analysis requires the absolute convergence of the double series obtained by expanding all the Bessel functions involved in the Kapteyn series into power series. He can consequently only prove his results when $|z| < 0{\cdot}659\ldots$, whereas they are really valid throughout the interior of the curve C; the smallest radius vector of this curve (when $\theta = \pm \frac{1}{2}\pi$) is $r = 0{\cdot}663\ldots$ and the largest (when $\theta = 0$ or π) is $r = 1$.

The theory of Kapteyn series has not, however, been exhausted by the mathematicians to whom reference has been made.

For example, the general description of the behaviour of $J_n(nx)$, when n is of any magnitude, whether large or not, and x varies from 0 to 1, deserves more investigation than it has yet received; it is possible to obtain some rather remarkable results by the use of Debye's contour integrals, and also by more elementary methods. It should be remarked that Debye's work is, to a certain extent, incomplete, as he has not published any investigation of the *remainders* in the expansions derived from his integrals, so that the precise ranges of the validity of his expansions

* A translation of Carlini's memoir (published at Milan, 1817) was given by Jacobi, *Astr. Nach.*, XXX (1850); *Ges. Werke*, VII, pp. 189–245.

† *Einleitung in die Theorie der Besselschen Funktionen* (Bern, 1898, 1900), I, pp. 96–107.

‡ *Philosophical Magazine*, Dec. 1907, Aug. 1908, Feb. 1910.

are by no means obvious.* Accordingly, in this paper, we shall only employ Debye's integrals and not the expansions derived from them; it may be remarked here that it is possible to derive certain inequalities (see § 10) from the integrals, which are not derivable from the expansions.

2. *Notation.*—In the Bessel functions $J_n(nx)$, $J_n(x)$, it is to be understood that x is real and that n is positive; but n is *not* restricted to be an integer, unless explicit mention is made of the fact or unless n occurs in such a summation as $\sum_{n=0}^{\infty}$; we use a prime to denote differentiation with regard to the argument of a function (*not* differentiation with regard to x); thus $J_n'(nx)$ means $\frac{1}{n}\frac{dJ_n(nx)}{dx}$.

3. *A definite integral for $J_n(nx)$, derived from Debye's first integral.*

We shall shew that Debye's analysis can be made to yield a definite integral for $J_n(nx)$, valid when $0 < x \leqslant 1$, in which the integrand is *positive*.

Take the ordinary contour integral

$$J_n(nx) = \frac{1}{2\pi i}\int_{-\infty}^{(0+)} t^{-n-1}e^{\frac{1}{2}nx(t-1/t)}\,dt,$$

in which $|\arg t| \leqslant \pi$. We follow Debye by choosing a contour† on which

$$\tfrac{1}{2}x\left(t-\frac{1}{t}\right)-\log t,$$

is purely real.

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* Debye makes the statement (*Math. Ann.*, LXVII, p. 543): "Der Beweis der Semikonvergenz gelingt leicht, indem man für den Rest der abgebrochenen, nach Potenzen von t fortschreitenden Reihe für $\Phi(t)$ eine ähnliche Ungleichung beachtet, wie sie z. B. von E. Borel, *Leçons sur les séries divergentes*, Paris, 1901, S. 25, nach dem Vorgange Cauchys zum Beweise der Semikonvergenz der Stirlingschen Reihe herangezogen wird."

I have failed to convince myself that $\Phi(t)$ satisfies an inequality of the type stated; but a discussion of the singularities of $\Phi(t)$ leads to conditions under which Debye's series are asymptotic in the sense of Poincaré (cf. a forthcoming paper on Parabolic Cylinder Functions). I understand that Mr. Littlewood has examined this problem very fully, but his results have not been published.

† Debye works with a slightly different integral, but the essence of his method lies in the choice of the contour of this type. The discussion of $\int e^{vf(s)}\phi(s)\,ds$, by choosing a contour on which $If(s)$ is constant, so that $Rf(s)$ falls off from its maxima as rapidly as possible (since $f(s)$ is analytic) is to be traced back to a posthumous paper by Riemann, *Werke* (1876), p. 405, and is known as the "Méthode du Col." The phrase "Method of steepest descents" might serve as an English equivalent.

Writing $$t = re^{i\theta} \qquad (-\pi \leqslant \theta \leqslant \pi),$$

we get $$\tfrac{1}{2}x \sin\theta \left(r + \frac{1}{r}\right) - \theta = 0,$$

and so we take $$r = \frac{\theta}{x \sin\theta}\{1 + \sqrt{(1 - x^2\theta^{-2}\sin^2\theta)}\}.$$

For the range of values of θ under consideration, r is real since $0 < x \leqslant 1$, and, as θ increases from $-\pi$ to $+\pi$, r decreases steadily from $+\infty$ to the minimum $x^{-1}\{1+\sqrt{(1-x^2)}\}$ at $\theta = 0$, and then increases steadily to $+\infty$; so that, if we write

$$t = \frac{\theta e^{i\theta}}{x \sin\theta}\{1 + \sqrt{(1 - x^2\theta^{-2}\sin^2\theta)}\},$$

then t describes a contour of the specified type.

Consequently, making the substitution, we have

$$J_n(nx) = \frac{1}{2\pi i}\int_{-\pi}^{\pi}\left[\exp\left\{\tfrac{1}{2}nx\left(r - \frac{1}{r}\right)\cos\theta - n\log r\right\}\right]\left(\frac{1}{r}\frac{dr}{d\theta} + i\right)d\theta.$$

Now r is an even function of θ, and so the real part of the expression under the integral sign is an odd function of θ, while the imaginary part is an even function of θ. And so we have at once

$$J_n(nx) = \frac{1}{\pi}\int_0^{\pi}\exp\left\{\tfrac{1}{2}nx\left(r - \frac{1}{r}\right)\cos\theta - n\log r\right\}d\theta$$

$$= \frac{1}{\pi}\int_0^{\pi}\exp\left\{n\cot\theta\sqrt{(\theta^2 - x^2\sin^2\theta)} - n\log\frac{\theta + \sqrt{(\theta^2 - x^2\sin^2\theta)}}{x\sin\theta}\right\}d\theta;$$

it is to be noted that, when $0 \leqslant \theta \leqslant \pi$,

$$+\theta\sqrt{(1 - x^2\theta^{-2}\sin^2\theta)} = +\sqrt{(\theta^2 - x^2\sin^2\theta)}.$$

For brevity we shall invariably write

$$\cot\theta\sqrt{(\theta^2 - x^2\sin^2\theta)} - \log\frac{\theta + \sqrt{(\theta^2 - x^2\sin^2\theta)}}{x\sin\theta} \equiv -F(\theta, x),$$

so that our result is that, when $0 < x \leqslant 1$,

$$J_n(nx) = \frac{1}{\pi}\int_0^{\pi} e^{-nF(\theta, x)}\,d\theta.$$

We observe that

$$\frac{d}{d\theta}F(\theta, x) = \frac{(1 - \theta\cot\theta)^2}{\sqrt{(\theta^2 - x^2\sin^2\theta)}} + \sqrt{(\theta^2 - x^2\sin^2\theta)} \geqslant 0,$$

and that $$\frac{d}{dx}F(0, x) = -x^{-1}\sqrt{(1-x^2)} \leqslant 0,$$

and hence $$F(\theta, x) \geqslant F(0, x) \geqslant F(0, 1) = 0,$$

so that $$F(\theta, x) \geqslant 0.$$

We also observe that

$$\frac{d}{dx}F(\theta, x) = -\frac{\theta - x^2 \sin\theta\cos\theta}{x\sqrt{(\theta^2 - x^2\sin^2\theta)}} \leqslant 0,$$

when $0 < x \leqslant 1$ and $0 \leqslant \theta \leqslant \pi$.

A property of $F(\theta, x)$ which will be required in § 6 is that, when $0 \leqslant \theta \leqslant \pi$,

$$F(\theta, x) \geqslant F(0, x) + \tfrac{1}{2}(\theta^2 - x^2\sin^2\theta)/\sqrt{(1+x^2)}.$$

We first prove that, when $0 \leqslant \theta \leqslant \pi$,

$$g(\theta, x) \equiv \frac{\theta - x^2\sin\theta\cos\theta}{\sqrt{(\theta^2 - x^2\sin^2\theta)}} \leqslant \sqrt{(1+x^2)}.$$

For $g(0, x)$ and $g(\pi, x)$ are equal to $\sqrt{(1-x^2)}$ and 1, which are less than $\sqrt{(1+x^2)}$; if, however, $g(\theta, x)$ had its greatest value not at an end-point, but at an *internal* point θ_0 of the range $(0, \pi)$, then

$$\frac{1 - x^2\cos 2\theta_0}{(\theta_0^2 - x^2\sin^2\theta_0)^{\frac{1}{2}}} - \frac{(\theta_0 - x^2\sin\theta_0\cos\theta_0)^2}{(\theta_0^2 - x^2\sin^2\theta_0)^{\frac{3}{2}}} = 0,$$

and then $$g(\theta, x) \leqslant g(\theta_0, x) = \sqrt{(1 - x^2\cos 2\theta_0)} \leqslant \sqrt{(1+x^2)},$$

so that, *in either case*, $$g(\theta, x) \leqslant \sqrt{(1+x^2)}.$$

Now $$\frac{dF(\theta, x)}{d\theta} \geqslant (\theta^2 - x^2\sin^2\theta)^{\frac{1}{2}} \geqslant (\theta - x^2\sin\theta\cos\theta)/\sqrt{(1+x^2)},$$

and so $$F(\theta, x) - F(0, x) = \int_0^\theta \frac{dF(\theta, x)}{d\theta}\,d\theta \geqslant \int_0^\theta (\theta - x^2\sin\theta\cos\theta)(1+x^2)^{-\frac{1}{2}}\,d\theta$$

$$= \tfrac{1}{2}(\theta^2 - x^2\sin^2\theta)/\sqrt{(1+x^2)},$$

which is the required result.

4. *A definite integral for* $J_n'(nx)$, *when* $0 < x \leqslant 1$.

By differentiating under the integral sign (a procedure which is easily

justified) we obtain, from the integral of § 3, the result that

$$J_n'(nx) = \frac{1}{\pi}\int_0^\pi e^{-nF(\theta,x)}\,\frac{\theta - x^2\sin\theta\cos\theta}{x\sqrt{(\theta^2 - x^2\sin^2\theta)}}\,d\theta.$$

This result may also be obtained by applying the method of § 3 to the formula

$$J_n'(nx) = \frac{1}{2\pi i}\int_{-\infty}^{(0+)} t^{-n-1}\,e^{\frac{1}{2}nx(t-1/t)}\,\tfrac{1}{2}\left(t - \frac{1}{t}\right)dt.$$

5. *A definite integral for* $\int_0^x J_n(ny)\,dy$, *when* $0 < x < 1$.

By integrating the ordinary contour integral for the Bessel function, we get

$$\int_0^x J_n(ny)\,dy = \frac{1}{2\pi i}\int_{-\infty}^{(0+)} \left\{e^{\frac{1}{2}nx(t-1/t)} - 1\right\}\left\{\tfrac{1}{2}n\left(t - \frac{1}{t}\right)\right\}^{-1} t^{-n-1}\,dt.$$

Take the path of integration to enclose the points ± 1; then

$$\int_{-\infty}^{(0+)} (t^2-1)^{-1}\,t^{-n}\,dt = 0,$$

for the path of integration may be deformed into a circle of indefinitely great radius, and on this circle the integrand is $O(t^{-2})$ at most.

Hence

$$\int_0^x J_n(ny)\,dy = \frac{1}{n\pi i}\int_{-\infty}^{(0+)} e^{\frac{1}{2}nx(t-1/t) - n\log t}\,(t^2-1)^{-1}\,dt,$$

if the path of integration encloses the points $t = \pm 1$. The contour of § 3 is therefore a possible one when $x < 1$; and making the substitution employed in § 3, we find that

$$\int_0^x J_n(ny)\,dy = \frac{x}{n\pi}\int_0^\pi e^{-nF(\theta,x)}\,\frac{\{\cot\theta\,(\theta^2 - x^2\sin^2\theta) - \theta(1-\theta\cot\theta)\}}{(\theta^2 - x^2\sin^2\theta\cos^2\theta)\sqrt{(\theta^2 - x^2\sin^2\theta)}}\,\sin^2\theta\,d\theta.$$

6. *Properties of* $J_n(nx)$ *and* $J_n'(nx)$, *regarded as functions of* n.

We shall first prove that $J_n(nx)$ and $J_n'(nx)$ are *decreasing functions* of n, when $0 < x \leqslant 1$. These results are obvious by differentiating the integrals of §§ 3, 4 under the integral sign.

Thus $$\frac{dJ_n(nx)}{dn} = -\frac{1}{\pi}\int_0^\pi F(\theta,x)\,e^{-nF(\theta,x)}\,d\theta \leqslant 0,$$

which gives one of the results; and the other follows in a similar manner. Since

$$F(\theta, x)-F(0, x) \geqslant 0,$$

it may be proved in the same way that

$$e^{nF(0,x)}\, J_n(nx) \quad \text{and} \quad e^{nF(0,x)}\, J_n'(nx)$$

are decreasing functions of n.

Next, since

$$\frac{d}{d\theta} F(\theta, x) \geqslant \sqrt{(\theta^2-x^2\sin^2\theta)} \geqslant \theta\sqrt{(1-x^2)},$$

we have, on integrating,

$$F(\theta, x)-F(0, x) \geqslant \tfrac{1}{2}\theta^2\sqrt{(1-x^2)}.$$

Hence
$$J_n(nx) \leqslant \frac{1}{\pi} e^{-nF(0,x)} \int_0^\pi \exp\left\{-\tfrac{1}{2}n\theta^2\sqrt{(1-x^2)}\right\} d\theta$$

$$< \frac{1}{\pi} e^{-nF(0,x)} \int_0^\infty \exp\left\{-\tfrac{1}{2}n\theta^2\sqrt{(1-x^2)}\right\} d\theta$$

$$= e^{-nF(0,x)}\, (2\pi n)^{-\frac{1}{2}}\, (1-x^2)^{-\frac{1}{4}},$$

that is to say
$$J_n(nx) < \frac{x^n e^{n\sqrt{(1-x^2)}}}{\{1+\sqrt{(1-x^2)}\}^n\, (2\pi n)^{\frac{1}{2}}\, (1-x^2)^{\frac{1}{4}}},$$

The expression on the right is the dominant term of Debye's expansion,* and consequently it gives a close approximation to $J_n(nx)$.

To obtain the similar inequality for $J_n'(nx)$, we write

$$\theta^2-x^2\sin^2\theta \equiv G(\theta, x),$$

and use the inequality (§ 3)

$$F(\theta, x) \geqslant F(0, x)+\tfrac{1}{2}G(\theta, x)(1+x^2)^{-\frac{1}{2}}.$$

Since we have (§ 4)

$$2xJ_n'(nx) = \frac{1}{\pi}\int_0^\pi e^{-nF(\theta,x)}\, \frac{dG(\theta, x)}{d\theta}\, \{G(\theta, x)\}^{-\frac{1}{2}}\, d\theta,$$

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* Debye actually gives an expression which is twice this; but he seems to have taken the integral of § 3 as $J_n(x) = \frac{1}{\pi i}\int_{-\infty}^{(0+)}$ instead of $\frac{1}{2\pi i}\int_{-\infty}^{(0+)}$. The expression here given is in agreement with Carlini's formula. It should be observed that Debye's expansion only gives this inequality for *sufficiently large* values of n, not for *all* positive values of n.

we get $$2xJ_n'(nx) \leqslant \frac{1}{\pi}\int_0^{\pi^2} G^{-\frac{1}{2}} \exp\{-nF(0,x)-\tfrac{1}{2}n(1+x^2)^{-\frac{1}{2}}G\}\,dG,$$

on taking $G(\theta, x)$ as a new variable G. On increasing the integral by taking ∞ instead of π^2 as the upper limit, we get

$$J_n'(nx) < \frac{x^{n-1}(1+x^2)^{\frac{1}{4}}\, e^{n\sqrt{(1-x^2)}}}{\{1+\sqrt{(1-x^2)}\}^n (2\pi n)^{\frac{1}{2}}}.$$

The absence of the factor $(1-x^2)^{\frac{1}{4}}$ from the denominator is worthy of note. Since

$$\operatorname*{Lim}_{\theta\to 0} \frac{\cot\theta(\theta^2-x^2\sin^2\theta)-\theta(1-\theta\cot\theta)}{(\theta^2-x^2\sin^2\theta\cos^2\theta)\sqrt{(\theta^2-x^2\sin^2\theta)}}\sin^2\theta = \frac{1}{\sqrt{(1-x^2)}},$$

it is probable (cf. § 5) that the dominant term of $\int_0^{\infty} J_n(ny)\,dy$ is

$$\frac{x^{n+1}e^{n\sqrt{(1-x^2)}}}{\{1+\sqrt{(1-x^2)}\}^n (2\pi n^3)^{\frac{1}{2}}(1-x^2)^{\frac{3}{4}}},$$

though a rigorous proof of this would be tedious and uninteresting.

7. *Some trigonometrical inequalities.*

The following inequalities are useful in discussing the properties of $F(\theta, x)$ and $F(\theta, 1)$; they are true when $0 \leqslant \theta \leqslant \pi$, and are proved by observing that the functions on the left all have positive differential coefficients* and all vanish when $\theta = 0$.

(I) $\theta+\sin\theta\cos\theta-2\theta^{-1}\sin^2\theta \geqslant 0$,

(II) $\theta+\sin\theta\cos\theta-2\theta^2\cot\theta \geqslant 0$,

(III) $\sin\theta-\theta\cos\theta-\frac{1}{3}\sin^3\theta \geqslant 0$.

We can deduce two weaker inequalities than (III), namely,

(IV) $\theta^2\operatorname{cosec}^2\theta-\theta^3\cot\theta\operatorname{cosec}^2\theta-\frac{1}{3}\sin^2\theta \geqslant 0$,

(V) $\theta^2\operatorname{cosec}^2\theta-1-\frac{1}{3}\sin^2\theta \geqslant 0$.

For the left-hand sides may be written in the forms

$$(\theta^2\operatorname{cosec}^2\theta-1)(1-\theta\cot\theta)+\operatorname{cosec}\theta(\sin\theta-\theta\cos\theta-\tfrac{1}{3}\sin^3\theta),$$

$$\theta\operatorname{cosec}^2\theta(\theta+\sin\theta\cos\theta-2\theta^{-1}\sin^2\theta)+\operatorname{cosec}\theta(\sin\theta-\theta\cos\theta-\tfrac{1}{3}\sin^3\theta),$$

* The differential coefficients are respectively

(I) $2(\cos\theta-\theta^{-1}\sin\theta)^2$, (II) $2(\cos\theta-\theta\operatorname{cosec}\theta)^2$, and (III) $\sin\theta(\theta-\sin\theta\cos\theta)$.

respectively; and the inequalities IV and V are then obvious from those already obtained.

8. *An inequality connected with* $F(\theta, x)$.

We shall next prove that, when $0 \leqslant x \leqslant 1$ and $0 \leqslant \theta \leqslant \pi$, then

$$\frac{dF(\theta, x)}{d\theta} - \{F(\theta, x) - F(0, x)\} \frac{\theta - x^2 \sin\theta \cos\theta}{\theta^2 - x^2 \sin^2\theta} \geqslant 0.$$

It will appear later (§ 12) that this inequality is instrumental in proving a very important theorem concerning the rate of increase of $J_n(nx)$.

It will first be shewn that

$$\frac{dF(\theta, x)}{d\theta} \Big/ \frac{d\sqrt{(\theta^2 - x^2 \sin^2\theta)}}{d\theta}$$

is a non-decreasing function of θ; that is to say that

$$\frac{(1 - \theta \cot\theta)^2 + \theta^2 - x^2 \sin^2\theta}{\theta - x^2 \sin\theta \cos\theta}$$

is a non-decreasing function of θ. This follows from the fact that the differential coefficient of the last expression with regard to θ is

$$\begin{aligned}(\theta - x^2 \sin\theta \cos\theta)^{-2} \big[&(\theta^2 \operatorname{cosec}^2\theta - 1 - \tfrac{1}{3}\sin^2\theta)(1 - x^2) \\ &+ 2(\theta^2 \operatorname{cosec}^2\theta - \theta^3 \cot\theta \operatorname{cosec}^2\theta - \tfrac{1}{3}\sin^2\theta)(1 - x^2) \\ &+ 2x^2(1 - \theta\cot\theta)(\theta \operatorname{cosec}\theta - \cos\theta)^2 + \sin^2\theta(1 - x^2)^2 \big],\end{aligned}$$

and every group of terms in this expression is zero or positive, by the results of § 7.

Writing, as usual, $\theta^2 - x^2 \sin^2\theta \equiv G(\theta, x)$, we have thus proved the preliminary theorem that

$$\frac{d}{d\theta}\left\{\frac{dF}{d\theta} \Big/ \frac{d\sqrt{G}}{d\theta}\right\} \geqslant 0.$$

Writing H for $\sqrt{G}$ and denoting differential coefficients with regard to θ by primes, we deduce the inequality

$$F''H' - F'H'' \geqslant 0.$$

Now $$\frac{d}{d\theta}\{(F'H/H') - F\} = \{F''H' - F'H''\} H/H'^2 \geqslant 0$$

by the inequality just proved.

Consequently $(F'H/H')-F'$ is a non-decreasing function of θ; and since it is equal to $-F(0,x)$ when θ is zero,* we have

$$\left\{\frac{dF(\theta,x)}{d\theta}H(\theta,x)\div\frac{dH(\theta,x)}{d\theta}\right\}-F(\theta,x)\geqslant -F(0,x).$$

Since $H/H'\geqslant 0$, this inequality at once gives

$$\frac{dF(\theta,x)}{d\theta}-\{F(\theta,x)-F(0,x)\}\frac{\theta-x^2\sin\theta\cos\theta}{\theta^2-x^2\sin^2\theta}\geqslant 0,$$

which is the inequality we have been seeking to establish.

9. *Properties of* $F(\theta,1)$.

We shall first shew that, when $0\leqslant\theta\leqslant\pi$, then

$$\frac{d}{d\theta}\left\{\frac{1}{\theta^2}\frac{dF(\theta,1)}{d\theta}\right\}\geqslant 0.$$

It is obvious that

$$\frac{1}{\theta^2}\frac{dF(\theta,1)}{d\theta}\equiv\frac{\{(1-\theta\cot\theta)/\theta^2\}^2}{\theta^{-2}\sqrt{(\theta^2-\sin^2\theta)}}+\theta^{-2}\sqrt{(\theta^2-\sin^2\theta)}$$

$$\equiv(U^2/V)+V, \text{ say.}$$

Now
$$\frac{dU}{d\theta}=\frac{\theta^2\operatorname{cosec}^2\theta+\theta\cot\theta-2}{\theta^3}$$

$$=W/\theta^3, \text{ say;}$$

and
$$\frac{dV}{d\theta}=-W\sin^2\theta/[\theta^3(\theta^2-\sin^2\theta)^{\frac{1}{2}}].$$

Consequently,

$$\frac{d}{d\theta}\left\{\frac{1}{\theta^2}\frac{dF(\theta,1)}{d\theta}\right\}=\frac{d\{(U^2/V)+V\}}{d\theta}$$

$$=\frac{W(\theta-\sin\theta\cos\theta)}{\theta^3(\theta^2-\sin^2\theta)^{\frac{3}{2}}}[\theta+\sin\theta\cos\theta-2\theta^2\cot\theta].$$

Now $W\geqslant 0$ by § 7 (I) and the remaining factors of the last expression are not negative [cf. § 7 (II)]; and so

$$\frac{d}{d\theta}\left\{\frac{1}{\theta^2}\frac{dF(\theta,1)}{d\theta}\right\}\geqslant 0,$$

which is the required inequality.

* Both $F'(\theta,x)$ and $H(\theta,x)/H'(\theta,x)\equiv(\theta^2-x^2\sin^2\theta)/(\theta-x^2\sin\theta\cos\theta)$ vanish when $\theta=0$.

This result may be written

$$\theta F''(\theta, 1) - 2F'(\theta, 1) \geqslant 0,$$

primes denoting differentiation with regard to θ.

That is to say $$\frac{d}{d\theta}\{\theta F'(\theta, 1) - 3F(\theta, 1)\} \geqslant 0.$$

The expression in brackets vanishes when $\theta = 0$, and hence, when $0 \leqslant \theta \leqslant \pi$, we have

$$\theta F'(\theta, 1) - 3F(\theta, 1) \geqslant 0.$$

This is the condition that $F(\theta, 1)/\theta^3$ should be a non-decreasing function of θ.

Consequently,

$$\frac{F(\theta, 1)}{\theta^3} \geqslant \lim_{\theta \to 0} \frac{F(\theta, 1)}{\theta^3} = \lim_{\theta \to 0} \frac{F'(\theta, 1)}{3\theta^2} = 4/(9\sqrt{3}),$$

that is to say $$F(\theta, 1) \geqslant 4\theta^3/(9\sqrt{3}).$$

Lastly we shall need the result that

$$2(\theta^2 - \sin^2\theta) F'(\theta, 1) - 3(\theta - \sin\theta\cos\theta) F(\theta, 1) \geqslant 0.$$

The truth of this may be seen by writing the expression on the left in the form

$$(\theta^2 - 2\sin^2\theta + \theta\sin\theta\cos\theta) F'(\theta, 1) + (\theta - \sin\theta\cos\theta)\{\theta F'(\theta, 1) - 3F(\theta, 1)\},$$

in which each group of terms is positive [cf. § 7 (I)].

10. *Properties of $J_n(n)$ and $J_n'(n)$.*

It has already been seen (§ 6) that these functions are decreasing functions of n. We shall now shew that *both** $n^{\frac{1}{3}}J_n(n)$ *and* $n^{\frac{2}{3}}J_n'(n)$ *are steadily increasing functions of* n.

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* It is fairly well known that, when n is large,

$$J_n(n) \sim \frac{\Gamma(\frac{1}{3})}{\pi 2^{\frac{2}{3}} 3^{\frac{1}{6}} n^{\frac{1}{3}}} - \frac{2^{\frac{2}{3}} 3^{\frac{1}{6}} \Gamma(\frac{2}{3})}{140\pi n^{\frac{5}{3}}} + \ldots,$$

and that $$J_n'(n) \sim \frac{3^{\frac{1}{6}} \Gamma(\frac{2}{3})}{\pi 2^{\frac{1}{3}} n^{\frac{2}{3}}} + \ldots .$$

But it is evidently impossible to infer from the monotonic properties of the terms on the right similar properties for the functions on the left. It should be emphasized that, if two functions

(I) We have

$$\frac{d\{n^{\frac{1}{3}}J_n(n)\}}{dn} = \frac{n^{-\frac{2}{3}}}{3\pi}\int_0^\pi e^{-nF(\theta,1)}\,d\theta - \frac{n^{\frac{1}{3}}}{\pi}\int_0^\pi F(\theta,1)\,e^{-nF(\theta,1)}\,d\theta$$

$$= \frac{n^{-\frac{2}{3}}}{3\pi}\left[\theta e^{-nF(\theta,1)}\right]_0^\pi + \frac{n^{\frac{1}{3}}}{3\pi}\int_0^\pi \{\theta F'(\theta,1) - 3F(\theta,1)\}\,e^{-nF(\theta,1)}\,d\theta$$

$$> 0,$$

for the integrated part vanishes at each limit, and it has been seen (§ 9) that the integrand is positive.

It is now evident that $n^{\frac{1}{3}}J_n(n)$ is an increasing function of n; and, *for all positive values of* n, we have

$$n^{\frac{1}{3}}J_n(n) < \operatorname*{Lim}_{n\to\infty}\{n^{\frac{1}{3}}J_n(n)\}$$

$$= \Gamma(\tfrac{1}{3})/\{\pi 2^{\frac{2}{3}}3^{\frac{1}{6}}\}$$

$$= 0{\cdot}44731.$$

It may be observed that $J_1(1) = 0{\cdot}44005$ and $2J_3(8) = 0{\cdot}44691$, so that, when $n > 1$, $J_n(n)$ lies between $n^{-\frac{1}{3}}\times 0{\cdot}44005$ and $n^{-\frac{1}{3}}\times 0{\cdot}44731$, while, when $n > 8$, $J_n(n)$ lies between $n^{-\frac{1}{3}}\times 0{\cdot}44691$ and $n^{-\frac{1}{3}}\times 0{\cdot}44731$.

(II) By analysis similar to that just given, it may be shewn that

$$\frac{d\{n^{\frac{2}{3}}J_n'(n)\}}{dn} = \frac{2n^{-\frac{1}{3}}}{3\pi}\left[e^{-nF(\theta,1)}\sqrt{(\theta^2-\sin^2\theta)}\right]_0^\pi$$

$$+ \frac{n^{\frac{2}{3}}}{3\pi}\int_0^\pi \left\{2\sqrt{(\theta^2-\sin^2\theta)}\,F'(\theta,1) - \frac{3(\theta-\sin\theta\cos\theta)}{\sqrt{(\theta^2-\sin^2\theta)}}F(\theta,1)\right\}e^{-nF(\theta,1)}\,d\theta$$

$$> 0,$$

by § 9, exactly as in case (I).

We also have $$n^{-\frac{1}{3}}\times 0{\cdot}32515 \leqslant J_n'(n) < n^{-\frac{1}{3}}\times 0{\cdot}41085 \quad (n \geqslant 1),$$

$$n^{-\frac{1}{3}}\times 0{\cdot}38854 \leqslant J_n'(n) < n^{-\frac{1}{3}}\times 0{\cdot}41085 \quad (n \geqslant 8).$$

11. *The function* $n^{\frac{1}{3}}J_n'(n)/J_n(n)$ *is a steadily increasing function of* n. This theorem, which is rather deeper than the theorems of § 10, may

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f, ϕ possess the property $f \sim \phi$ and if ϕ *steadily increases* (decreases) nothing can be inferred as to the *steadiness* of the increase (decrease) of f, without some further knowledge of the properties of f and ϕ. I may remark that rigorous proofs of the asymptotic formulæ for $J_n(n)$ and $J_n'(n)$ just stated can be obtained without the use of Debye's methods; these proofs are given in the *Proceedings of the Cambridge Philosophical Society*, Vol. XIX (1917), pp. 42–48.

be regarded as leading up to the more difficult theorem given in § 12; it is of some importance in connexion with the maxima of $J_n(nx)$.

To prove it, we shall make use of some evident properties of repeated integrals :—a method equally effective in proving the theorem of § 12.

Making use of the four results, already given in § 10,

$$n^{\frac{1}{3}}J_n(n) = \frac{n^{\frac{1}{3}}}{\pi}\int_0^\pi e^{-nF(\psi,\,1)}\,d\psi,$$

$$n^{\frac{2}{3}}J_n'(n) = \frac{n^{\frac{2}{3}}}{\pi}\int_0^\pi e^{-nF(\theta,\,1)}\,\frac{\theta-\sin\theta\cos\theta}{\sqrt{(\theta^2-\sin^2\theta)}}\,d\theta,$$

$$\frac{d}{dn}\{n^{\frac{1}{3}}J_n(n)\} = \frac{n^{\frac{1}{3}}}{\pi}\int_0^\pi e^{-nF(\psi,\,1)}\{\tfrac{1}{3}\psi F'(\psi,\,1)-F(\psi,\,1)\}\,d\psi,$$

$$\frac{d}{dn}\{n^{\frac{2}{3}}J_n'(n)\} = \frac{n^{\frac{2}{3}}}{\pi}\int_0^\pi e^{-nF(\theta,\,1)}\left\{\tfrac{2}{3}\sqrt{(\theta^2-\sin^2\theta)}\,F'(\theta,\,1)\right.$$
$$\left.-\frac{\theta-\sin\theta\cos\theta}{\sqrt{(\theta^2-\sin^2\theta)}}\,F(\theta,\,1)\right\}d\theta,$$

we see that

$$\{n^{\frac{1}{3}}J_n(n)\}^2\frac{d}{dn}\left\{\frac{n^{\frac{2}{3}}J_n'(n)}{n^{\frac{1}{3}}J_n(n)}\right\} = \frac{n}{\pi^2}\int_0^\pi\int_0^\pi \Omega(\theta,\psi)\,e^{-n\{F(\theta,\,1)+F(\psi,\,1)\}}\,d\theta\,d\psi,$$

where
$$\begin{aligned}\Omega(\theta,\psi) &= \tfrac{2}{3}\sqrt{(\theta^2-\sin^2\theta)}\,F'(\theta,1)-\frac{\theta-\sin\theta\cos\theta}{\sqrt{(\theta^2-\sin^2\theta)}}\,F(\theta,1)\\ &\quad-\frac{\theta-\sin\theta\cos\theta}{\sqrt{(\theta^2-\sin^2\theta)}}\{\tfrac{1}{3}\psi F'(\psi,1)-F(\psi,1)\}\\ &= 2\theta^{-1}\sqrt{(\theta^2-\sin^2\theta)}\{\tfrac{1}{3}\theta F'(\theta,1)-F(\theta,1)\}\\ &\quad+\theta^{-1}(\theta^2-\sin^2\theta)^{-\frac{1}{2}}\{\theta^2+\theta\sin\theta\cos\theta-2\sin^2\theta\}\,F(\theta,1)\\ &\quad-\frac{\theta-\sin\theta\cos\theta}{\sqrt{(\theta^2-\sin^2\theta)}}\{\tfrac{1}{3}\psi F'(\psi,1)-F(\psi,1)\}\\ &\geqslant 2\theta^{-1}\sqrt{(\theta^2-\sin^2\theta)}\{\tfrac{1}{3}\theta F'(\theta,1)-F(\theta,1)\}\\ &\quad-\frac{\theta-\sin\theta\cos\theta}{\sqrt{(\theta^2-\sin^2\theta)}}\{\tfrac{1}{3}\psi F'(\psi,1)-F(\psi,1)\}.\end{aligned}$$

By interchanging the parametric variables θ and ψ, it is evident that we have

$$2\{n\,J_n(n)\}^2\frac{d}{dn}\left\{\frac{n^{\frac{2}{3}}J_n'(n)}{n^{\frac{1}{3}}J_n(n)}\right\}$$
$$= \frac{n}{\pi^2}\int_0^\pi\int_0^\pi\{\Omega(\theta,\psi)+\Omega(\psi,\theta)\}\,e^{-n\{F(\theta,\,1)+F(\psi,\,1)\}}\,d\theta\,d\psi.$$

To obtain the required result it is obviously sufficient to prove that $\Omega(\theta, \psi)+\Omega(\psi, \theta)$ is never negative, for then $n^{\frac{1}{3}}J_n'(n)/J_n(n)$ will have a positive differential coefficient. To this proof we now proceed.

From the inequality which has already been obtained for Ω it is evident that

$$\begin{aligned}
&\Omega(\theta, \psi)+\Omega(\psi, \theta)\\
&\geqslant \left\{\frac{2\sqrt{(\theta^2-\sin^2\theta)}}{\theta}-\frac{\psi-\sin\psi\cos\psi}{\sqrt{(\psi^2-\sin^2\psi)}}\right\}\{\tfrac{1}{3}\theta F'(\theta, 1)-F(\theta, 1)\}\\
&\quad+\left\{\frac{2\sqrt{(\psi^2-\sin^2\psi)}}{\psi}-\frac{\theta-\sin\theta\cos\theta}{\sqrt{(\theta^2-\sin^2\theta)}}\right\}\{\tfrac{1}{3}\psi F'(\psi, 1)-F(\psi, 1)\}\\
&= \left\{\frac{2\sqrt{(\theta^2-\sin^2\theta)}}{\theta}-\frac{2\sqrt{(\psi^2-\sin^2\psi)}}{\psi}\right\}[\{\tfrac{1}{3}\theta F'(\theta, 1)-F(\theta, 1)\}\\
&\qquad\qquad-\{\tfrac{1}{3}\psi F'(\psi, 1)-F(\psi, 1)\}]\\
&\quad+\left\{\frac{2\sqrt{(\psi^2-\sin^2\psi)}}{\psi}-\frac{\psi-\sin\psi\cos\psi}{\sqrt{(\psi^2-\sin^2\psi)}}\right\}\{\tfrac{1}{3}\theta F'(\theta, 1)-F(\theta, 1)\}\\
&\quad+\left\{\frac{2\sqrt{(\theta^2-\sin^2\theta)}}{\theta}-\frac{\theta-\sin\theta\cos\theta}{\sqrt{(\theta^2-\sin^2\theta)}}\right\}\{\tfrac{1}{3}\psi F'(\psi, 1)-F(\psi, 1)\}.
\end{aligned}$$

We shall shew that each of these three terms is positive.

It is evident that $\sqrt{(1-\theta^{-2}\sin^2\theta)}$ is an increasing function, while $\frac{1}{3}\theta F'(\theta, 1)-F(\theta, 1)$ is known to be a positive increasing function (§ 9); so the factors of the first term are both positive or both negative according as $\theta \gtrless \psi$; and the second factors of the second and third terms in the expression are both positive; the first factors in these terms are also positive, since

$$\frac{2\sqrt{(\theta^2-\sin^2\theta)}}{\theta}-\frac{\theta-\sin\theta\cos\theta}{\sqrt{(\theta^2-\sin^2\theta)}}=\frac{\theta^2-2\sin^2\theta+\theta\sin\theta\cos\theta}{\theta\sqrt{(\theta^2-\sin^2\theta)}},$$

and this is positive by § 7 (I).

Since all three terms have now been shewn to be positive, it follows that $\Omega(\theta, \psi)+\Omega(\psi, \theta)$ is positive; and it has already been seen that this is a sufficient condition for $n^{\frac{1}{3}}J_n'(n)/J_n(n)$ to be an increasing function.

It may be shewn that

$$0{\cdot}73889 \leqslant n^{\frac{1}{3}}J_n'(n)/J_n(n) < 0{\cdot}91849 \quad (n \geqslant 1),$$

$$0{\cdot}86939 \leqslant n^{\frac{1}{3}}J_n'(n)/J_n(n) < 0{\cdot}91849 \quad (n \geqslant 8).$$

12. *The cardinal theorem concerning $J_n(nx)$ qua function of n.*

We shall now prove the important theorem that, *whenever* $n > 0$ *and* $0 \leqslant x \leqslant 1$, *then* $J_n(nx)/J_n(n)$ *is a non-increasing function of* n.

This theorem is of vital importance in the investigation of the uniformity of the convergence of a Kapteyn series near $x = 1$.

[We shall prove the theorem only for the range $\delta \leqslant x \leqslant 1$, where δ is any positive number; the proof fails for the range $0 \leqslant x \leqslant \delta$, on account of the presence of an x in some denominators; but it is obvious when $0 \leqslant x \leqslant \delta$, since $e^{nF(0,x)}J_n(nx)$ and $e^{-nF(0,x)}/J_n(n)$ are non-increasing functions when δ is sufficiently small; the importance of the theorem lies in the fact that it is true *for values of* x *in the neighbourhood of* $x = 1$.]

It will first be shewn that

$$J_n(nx)\frac{\partial^2 J_n(nx)}{\partial n\,\partial x} - \frac{\partial J_n(nx)}{\partial x}\,\frac{\partial J_n(nx)}{\partial n} \geqslant 0.$$

With the usual notation, we have

$$J_n(nx) = \frac{1}{\pi}\int_0^\pi e^{-nF(\psi,x)}\,d\psi,$$

$$\frac{\partial J_n(nx)}{\partial x} = \frac{n}{2\pi x}\int_0^\pi \{G(\theta,x)\}^{-\frac{1}{2}}\frac{dG(\theta,x)}{d\theta}\,e^{-nF(\theta,x)}\,d\theta,$$

and, by differentiating under the integral sign,

$$\frac{\partial J_n(nx)}{\partial n} = -\frac{1}{\pi}\int_0^\pi F(\psi,x)\,e^{-nF(\psi,x)}\,d\psi,$$

$$\frac{\partial^2 J_n(nx)}{\partial n\,\partial x} = \frac{1}{2\pi x}\int_0^\pi \{G(\theta,x)\}^{-\frac{1}{2}}\frac{dG(\theta,x)}{d\theta}\,e^{-nF(\theta,x)}\,d\theta$$

$$-\frac{n}{2\pi x}\int_0^\pi F(\theta,x)\{G(\theta,x)\}^{-\frac{1}{2}}\frac{dG(\theta,x)}{d\theta}\,e^{-nF(\theta,x)}\,d\theta$$

$$= \frac{n}{\pi x}\int_0^\pi\Big[\{G(\theta,x)\}^{\frac{1}{2}}\frac{dF(\theta,x)}{d\theta} - \tfrac{1}{2}F(\theta,x)\{G(\theta,x)\}^{-\frac{1}{2}}\frac{dG(\theta,x)}{d\theta}\Big]$$

$$\times e^{-nF(\theta,x)}\,d\theta,$$

on integrating by parts the former of the two integrals.

Consequently

$$J_n(nx)\frac{\partial^2 J_n(nx)}{\partial n\,\partial x} - \frac{\partial J_n(nx)}{\partial n}\,\frac{\partial J_n(nx)}{\partial x} = \frac{n}{2\pi^2 x}\int_0^\pi\int_0^\pi \Omega_1(\theta,\psi)\,e^{-n\{F(\theta,x)+F(\psi,x)\}}\,d\theta\,d\psi,$$

where

$$\Omega_1(\theta, \psi) = 2\{G(\theta, x)\}^{\frac{1}{2}} \frac{dF(\theta, x)}{d\theta} + \{F(\psi, x) - F(\theta, x)\}\{G(\theta, x)\}^{-\frac{1}{2}} \frac{dG(\theta, x)}{d\theta}$$

$$\geqslant 2\{G(\theta, x)\}^{\frac{1}{2}} \left[\frac{dF(\theta, x)}{d\theta} + \{F(0, x) - F(\theta, x)\} \frac{\theta - x^2 \sin\theta \cos\theta}{\theta^2 - x^2 \sin^2\theta}\right],$$

since $$\{F(\psi, x) - F(0, x)\}\{G(\theta, x)\}^{-\frac{1}{2}} \frac{dG(\theta, x)}{d\theta} \geqslant 0.$$

This inequality shews that $\Omega_1(\theta, \psi)$ is greater than or equal to an expression which has been shewn (§ 8) to be greater than or equal to zero; and so

$$\Omega_1(\theta, \psi) \geqslant 0.$$

From this result it is obvious that

$$J_n(nx) \frac{\partial^2 J_n(nx)}{\partial n \partial x} - \frac{\partial J_n(nx)}{\partial n} \frac{\partial J_n(nx)}{\partial x} \geqslant 0,$$

that is to say

$$\frac{\partial}{\partial x}\left\{\frac{\partial J_n(nx)}{\partial n} \Big/ J_n(nx)\right\} \geqslant 0.$$

This inequality shews that $\dfrac{\partial J_n(nx)}{\partial n} \Big/ J_n(nx)$ is a non-decreasing function of x so long as n remains unchanged; and so, since $x \leqslant 1$, we must have

$$\frac{\partial J_n(nx)}{\partial n} \Big/ J_n(nx) \leqslant \frac{\partial J_n(n)}{\partial n} \Big/ J_n(n).$$

But, since $J_n(nx)$ and $J_n(n)$ are both positive, the last inequality may be written

$$\frac{\partial}{\partial n}\{J_n(nx)/J_n(n)\} \leqslant 0;$$

and this exhibits the result which was to be proved, namely, that $J_n(nx)/J_n(n)$ is a non-increasing function of n.

13. *The behaviour of $J_n(x)$ as x increases from 0 to n.*

It is possible to obtain a fair amount of information about the behaviour of $J_n(x)$ from quite elementary considerations, without making use of the elaborate machinery provided by Debye's contour integrals.

We shall first shew that $J_n(x)$ *increases to a maximum* as x increases from 0 to M_1, where $M_1 > n$.*

* Since $J_n(x)$ has an infinity of real zeros, it follows from Rolle's theorem that it has an infinity of maxima and minima.

It is evident from the power series for $J_n'(x)$ that $J_n(x)$ is an increasing function for sufficiently small values of x. Accordingly let M_1 be the smallest positive value of x for which $J_n(x)$ is a maximum, so that $J_n'(M_1) = 0$ and $J_n'(x)$ changes sign from $+$ to $-$ at $x = M_1$; $J_n''(M_1)$ is therefore negative or zero. Hence from the equation

$$M_1^2 J_n''(M_1) + M_1 J_n'(M_1) + (M_1^2 - n^2) J_n(M_1) = 0,$$

we must have $M_1^2 - n^2 \geqslant 0$; if $M_1 = n$, so that $J_n''(M_1) = 0$, we should deduce from the equation

$$x^2 J_n'''(x) + 3x J_n''(x) + (1 + x^2 - n^2) J_n'(x) + 2x J_n(x) = 0,$$

that $J_n'''(n) < 0$, which would mean that $J_n(x)$ was decreasing (and not a maximum) at $x = n$. Hence *we cannot have* $M_1 = n$, and so we must have* $M_1 > n$.

In like manner, we deduce from the equation

$$x^2(x^2 - n^2) J_n'''(x) + x(x^2 - 3n^2) J_n''(x) + \{(x^2 - n^2)^2 - x^2 - n^2\} J_n'(x) = 0,$$

that, if $n \geqslant 1$, and if μ is the smallest value of x for which $J_n'(x)$ is a maximum, then

$$(\mu^2 - n^2)^2 - \mu^2 - n^2 < 0,$$

so that

$$\mu > \sqrt{\{n^2 + \tfrac{1}{2} - (2n^2 + \tfrac{1}{4})^{\frac{1}{2}}\}}.$$

Also $\mu < n$ because, by § 4, $J_n'(n) > 0$ and $J_n''(n) = -n^{-1} J_n'(n) < 0$, from the differential equation.

14. *The first maximum of* $J_n(x)$.

We shall now obtain (by elementary methods) rough approximations to M_1 and $J_n(M_1)$, where M_1 is defined as in § 13. From the inequalities of § 10, it will follow that $(M_1 - n)/n^{\frac{1}{3}}$ lies between positive bounds; so that M_1 lies in a range of values of x which forms a stage of transition between ranges of values in which the expansions of Debye are valid. The oscillatory factors, which multiply the pairs of integrals representing $J_n(x)$ when $x > n$ in Debye's analysis, render it extremely likely that in any investigation of M_1 and $J_n(M_1)$, by means of Debye's integrals, very elaborate analysis would be inevitable.

We first introduce a new variable ξ defined in terms of x by the equation

$$x = ne^{\xi},$$

and we write $\mu_1 \equiv \log(M_1/n)$, so that the first maximum of $J_n(ne^{\xi})$ for a positive value of ξ is at $\xi = \mu_1$.

* If $n = 1$, $M_1 = 1{\cdot}84$ (Gray and Matthews, *Bessel Functions*, p. 249).

From the equation

$$\frac{d^2J_n(x)}{d\xi^2}+n^2(e^{2\xi}-1)\,J_n(x)=0,$$

we see that $\frac{d^2J_n(x)}{d\xi^2}$ vanishes when $\xi=0$ and is negative in the range $0<\xi\leqslant\mu_1$; while from the equation

$$\frac{d^3J_n(x)}{d\xi^3}+n^2(e^{2\xi}-1)\frac{dJ_n(x)}{d\xi}+2n^2e^{2\xi}J_n(x)=0,$$

we see that, when $0\leqslant\xi\leqslant\mu_1$, then

$$\frac{d^3J_n(x)}{d\xi^3}\leqslant-2n^2e^{2\xi}J_n(x)<-2n^2J_n(n)<0.$$

From this inequality we see that $\frac{d^2J_n(x)}{d\xi^2}$ decreases steadily throughout the range $0\leqslant\xi\leqslant\mu_1$.

Next let a_0, a_1, a_2, a_3 be the values of $J_n(ne^{\xi})$ and its first three derivates with regard to ξ at $\xi=0$; so that

$$a_0=J_n(n),\qquad a_1=nJ_n'(n),$$

$$a_2=0,\qquad a_3=-2n^2J_n(n);$$

also let a_2' be the value of $\frac{d^2J_n(ne^{\xi})}{d\xi^2}$ at $\xi=\mu_1$, so that

$$a_2'=-(M_1^2-n^2)\,J_n(M_1).$$

Regarding (ξ,η) as Cartesian coordinates, let us plot the three curves

$$\eta_1=a_0+a_1\xi+\tfrac{1}{2}a_2\xi^2+\tfrac{1}{6}a_3\xi^3,$$

$$\eta_2=J_n(ne^{\xi}),$$

$$\eta_3=a_0+a_1\xi+\tfrac{1}{2}a_2'\xi^2,$$

from $\xi=0$ to $\xi=\mu_1$.

The cubic and the second curve have contact of the third order at $\xi=0$, while the second curve and the parabola touch.

Also $$\frac{d^3(\eta_1-\eta_2)}{d\xi^3}=a_3-\frac{d^3J_n(x)}{d\xi^3}\geqslant 0.$$

Integrating successively and noting that the second and first differential

coefficients of $\eta_1-\eta_2$ vanish at $\xi=0$, we get in turn

$$\frac{d\eta_1}{d\xi}\geqslant\frac{d\eta_2}{d\xi}, \quad \eta_1\geqslant\eta_2 \quad (0\leqslant\xi\leqslant\mu_1).$$

Hence the cubic is above the second curve and has a steeper slope in this range; and so the summit of the cubic* is above and on the right of the point $[\mu_1, J_n(M_1)]$.

That is to say

$$\mu_1<\sqrt{\left(\frac{2a_1}{-a_3}\right)}, \quad J_n(M_1)<a_0+\tfrac{2}{3}a_1\sqrt{\left(\frac{2a_1}{-a_3}\right)}.$$

In a similar manner, by using the inequality

$$\frac{d^2J_n(x)}{d\xi^2}\geqslant a_2'$$

(the expression on the left being a *decreasing* function), we find that the point $[\mu_1, J_n(M_1)]$ is above and on the right of the vertex of the parabola, so that

$$\mu_1>\frac{a_1}{-a_2'}, \quad J_n(M_1)>a_0-\tfrac{1}{2}a_1^2/a_2'.$$

On substituting for a_0, a_1, a_2, a_3, a_2' their values and writing u_n in place of $J_n'(n)/J_n(n)$ for brevity, we have

$$\sqrt{\left(\frac{u_n}{n}\right)}>\mu_1>\frac{nu_n}{(M_1^2-n^2)\,J_n(M_1)/J_n(n)}, \tag{i}$$

$$J_n(n)\left\{1+\frac{\frac{1}{2}n^2u_n^2}{(M_1^2-n^2)\,J_n(M_1)/J_n(n)}\right\}<J_n(M_1)<J_n(n)\left\{1+\tfrac{2}{3}n^{\frac{1}{2}}u_n^{\frac{3}{2}}\right\}. \tag{ii}$$

From these inequalities, some more useful ones can be obtained. For we have

$$\left\{\frac{M_1^2}{n^2}-1\right\}\mu_1>\frac{u_n/n}{J_n(M_1)/J_n(n)}>\frac{u_n/n}{1+\frac{2}{3}n^{\frac{1}{2}}u_n^{\frac{3}{2}}}.$$

On writing $$(u_n/n)\div\left\{1+\tfrac{2}{3}n^{\frac{1}{2}}u_n^{\frac{3}{2}}\right\}\equiv 2v_n^2,$$

we get $$\left\{\left(\frac{M_1}{n}\right)^2-1\right\}\log\left(\frac{M_1}{n}\right)>2v_n^2>v_n\log(1+2v_n)$$

$$=\{(2v_n+1)-1\}\log\sqrt{(1+2v_n)}.$$

---

* As ξ increases from 0 to $+\infty$, η_1 increases from a_0 to a maximum at $\xi=\sqrt{(-2a_1/a_3)}$, and then decreases to $-\infty$.

But $(X^2-1)\log X$ is an increasing function of X, and so we must have

$$(M_1/n) > \sqrt{(1+2v_n)}.$$

It now follows from (i) that we have

$$n\sqrt{(1+2v_n)} < M_1 < n\exp\{\sqrt{(u_n/n)}\}.$$

Since $n^{\frac{2}{3}}v_n$ and $n^{\frac{1}{3}}u_n$ tend to finite positive limits as $n\to\infty$ (see § 10, footnote), it follows from this inequality that $(M_1-n)/n^{\frac{1}{3}}$ has positive bounds as $n\to\infty$, so that M_1 lies in the transition-stage described at the beginning of this section.

The first part of the inequality (ii) does not yield information of any special interest.* But the second part of (ii), combined with § 11, shews that

$$J_n(M_1) < J_n(n)\{1+\operatorname*{Lim}_{n\to\infty}(\tfrac{2}{3}n^{\frac{1}{3}}u_n^{\frac{3}{2}})\}$$

$$= J_n(n)\times 1{\cdot}5868.$$

15. *The values of $J_n(x)$ when it is stationary.*

It will now be shewn that, if $M_1, M_2, M_3, \ldots$ are the positive values of x for which $J_n(x)$ is a maximum, while $m_1, m_2, m_3, \ldots$ are the positive values of x for which it is a minimum, so arranged that

$$M_1 < m_1 < M_2 < m_2 < M_3 < \ldots,$$

then $$J_n(M_1) > -J_n(m_1) > J_n(M_2) > -J_n(m_2) > \ldots > 0.$$

[It is obvious from this result that, for all real values of x,

$$|J_n(x)| \leqslant J_n(M_1).]$$

To establish the required set of inequalities, we write

$$J_n^2(x)+\{x^2J_n'^2(x)/(x^2-n^2)\} \equiv \Lambda(x),$$

and then $$\Lambda'(x) = -2x^3J_n'^2(x)/(x^2-n^2)^2,$$

and this is negative, except when $J_n(x)$ is stationary. Therefore

$$\Lambda(M_1) > \Lambda(m_1) > \Lambda(M_2) > \Lambda(m_2) > \ldots,$$

That is to say $$J_n^2(M_1) > J_n^2(m_1) > J_n^2(M_2) > J_n^2(m_2) > \ldots.$$

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* We evidently can shew that

$$J_n(M_1) > J_n(n)\left\{1+\frac{u_n^2n^2/(M_1^2-n^2)}{1+[1+2u_n^2n^2/(M_1^2-n^2)]^{\frac{1}{2}}}\right\},$$

whence $J_n(M_1) > J_n(n)\times 1{\cdot}1857$, when $n > 8$.

Now there are an odd number of zeros of $J_n'(x)$ between each consecutive pair of zeros of $J_n(x)$, and there are an odd number of zeros of

$$\frac{d}{dx}\{xJ_n'(x)\} \equiv (n^2-x^2)J_n(x)/x,$$

between each pair of zeros of $xJ_n'(x)$. Hence, between each consecutive pair of the numbers $M_1(>n)$, m_1, M_2, m_2, ... there is one and only one zero of $J_n(x)$; and so $J_n(M_r)>0$, $J_n(m_r)<0$. The required result is now evident by a use of the last sequence of inequalities.

16. *The stationary values of $J_n(x)$ for large values of the argument.*

The asymptotic formula of Hankel,* namely,

$$J_n(x) = \{2/(\pi x)\}^{\frac{1}{2}}\cos(x-\tfrac{1}{2}n\pi-\tfrac{1}{4}\pi)+O(x^{-\frac{3}{2}}),$$

suggests that we may be able to prove a definite inequality consistent with

$$J_n(M_r) = O(M_r^{-\frac{1}{2}}),$$

when r is large.

We shall, in fact, shew that:

(I) *The values assumed by $(x^2-n^2)^{\frac{1}{4}}|J_n(x)|$, when x takes the values M_1, m_1, M_2, m_2, ..., form an increasing sequence whose members are all less than $(2/\pi)$.*

(II) *The values assumed by $x^{\frac{1}{2}}|J_n(x)|$, when x takes the values M_r, m_r, M_{r+1}, m_{r+1}, ..., form a decreasing sequence whose members are all greater than $(2/\pi)$, provided that*†

(i) $n>\frac{1}{2}\sqrt{3}$, *and* (ii) $M_r^2>n^2\{4n^2+4+\sqrt{(48n^2+13)}\}/(4n^2-3)$.

We consider the function

$$A(x)J_n^2(x)+2B(x)J_n(x)J_n'(x)+C(x)J_n'^2(x) \equiv \Theta(x),$$

say, where $A(x)$, $B(x)$, $C(x)$ are to be suitably chosen.

We have

$$\begin{aligned}\Theta'(x) = &\{A'(x)-2(x^2-n^2)B(x)/x^2\}J_n^2(x)\\ &+2\{B'(x)+A(x)-B(x)/x-(x^2-n^2)C(x)/x^2\}J_n(x)J_n'(x)\\ &+\{C'(x)+2B(x)-2C(x)/x\}J_n'^2(x).\end{aligned}$$

* The associated formula $J_n'(x) = -\{2/(\pi x)\}^{\frac{1}{2}}\sin(x-\frac{1}{2}n\pi-\frac{1}{4}\pi)+O(x^{-\frac{3}{2}})$ is required later in this section.

† These restrictions are not required in case (I).

It is evident that $\Theta'(x)$ reduces to $D(x)\,J_n'^2(x)$, where

$$D(x) \equiv C'(x)+2B(x)-2C(x)/x,$$

provided that $A(x)$ be chosen arbitrarily, and that $B(x)$ and $C(x)$ are then defined by the equations:

$$2B(x) = x^2A'(x)/(x^2-n^2),$$

$$C(x) = x^2\{B'(x)+A(x)-B(x)/x\}/(x^2-n^2).$$

Case I.—If $A(x) \equiv (x^2-n^2)^{\frac{1}{2}}$, it is easily verified by elementary algebra that

$$2D(x)(x^2-n^2)^{\frac{3}{2}} \equiv x^3(3x^4+14x^2n^2+4n^4) > 0.$$

Hence, in this case, $\Theta(x)$ is an increasing function, and so the values of $\sqrt{\{\Theta(x)\}}$ when x takes the values M_1, m_1, M_2, m_2, ... form an increasing sequence whose members are all less than

$$\lim_{x\to\infty} \sqrt{\{\Theta(x)\}} = 2/\pi,$$

But, when x has any one of the values in question,

$$\sqrt{\{\Theta(x)\}} = (x^2-n^2)^{\frac{1}{4}}\,|J_n(x)|,$$

and the truth of theorem (I) is evident.

Case II.—If $A(x) \equiv x$, we find that

$$2D(x)(x^2-n^2)^4 = -x^2\{(4n^2-3)x^4-8n^2(n^2+1)x^2+n^4(4n^2-1)\},$$

and this is *negative* when $4n^2 > 3$ and x exceeds the greater positive root of the quartic

$$(4n^2-3)x^4-8n^2(n^2+1)x^2+n^4(4n^2-1) = 0.$$

This root is given by

$$x^2 = n^2\{4n^2+4+\sqrt{(48n^2+13)}\}/(4n^2-3).$$

Hence, in this case, $\Theta(x)$ is a *decreasing* function for the values of x under consideration, and by reasoning similar to that employed in Case I, we at once deduce the truth of theorem (II).

17. *The convergence and continuity of Kapteyn series.*

The convergence of the series

$$\sum_{n=0}^{\infty} a_n J_{\nu+n}\{(\nu+n)z\},$$

in the *complex* variable z, is not of any very great interest; for, if

$$\overline{\lim_{n\to\infty}} |\sqrt[n]{a_n}| = 1/\rho,$$

it follows from Debye's integrals (given in his second paper, in the case of complex variables*), that the series converges absolutely when

$$\left| \frac{ze^{\sqrt{(1-z^2)}}}{1+\sqrt{(1-z^2)}} \right| < \rho,$$

except possibly at $z = \pm 1$; and at points inside a domain throughout which this inequality is satisfied, the series represents an analytic function of z, with a branch-point at the origin, except when ν is an integer. This is a result complementary to that proved by Kapteyn, to which reference was made in § 1.

The convergence of the series along the real axis is much more interesting, for we can prove the following theorem of an Abelian type :—

If $\sum_{n=1}^{\infty} a_n n^{-\frac{1}{3}}$ *converges, then*

$$\sum_{n=0}^{\infty} a_n J_{\nu+n}\{(\nu+n)x\}$$

converges uniformly throughout the closed interval† $0 \leqslant x \leqslant 1$, *and it is therefore a continuous function of* x *in this interval; so that, in particular,*

$$\operatorname*{Lim}_{x\to 1-0} \sum_{n=0}^{\infty} a_n J_{\nu+n}\{(\nu+n)x\} = \sum_{n=0}^{\infty} a_n J_{\nu+n}(\nu+n).$$

[It is worthy of note that this is the best theorem of the kind which can be proved for the interval $0 \leqslant x \leqslant 1$. For, if $\sum_{n=1}^{\infty} a_n n^{-\frac{1}{3}}$ does not converge, the Kapteyn series *cannot* converge at $x = 1$; this will be seen in the course of the following analysis.]

The proof of this theorem is based on results proved earlier in this paper, and it presents no great difficulty.

We first observe that, since $\sum_{n=1}^{\infty} a_n n^{-\frac{1}{3}}$ converges, and $(\nu+n)^{-\frac{1}{3}} n^{\frac{1}{3}}$ is a bounded monotonic function of n, it follows from Abel's test for convergence that the series $\sum_{n=1}^{\infty} (a_n n^{-\frac{1}{3}})\{(\nu+n)^{-\frac{1}{3}} n^{\frac{1}{3}}\}$ is convergent.

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* An inequality somewhat similar to that of § 6 is valid.

† The uniformity of the convergence for the range $0 \leqslant x \leqslant 1-\delta$, where $\delta > 0$ is almost obvious from the inequality of § 6.

That is to say, $\sum_{n=1}^{\infty} a_n(\nu+n)^{-\frac{1}{3}}$ converges; hence $a_n(\nu+n)^{-\frac{1}{3}} \to 0$ as $n \to \infty$, and so $\sum_{n=1}^{\infty} a_n U_n$ converges (absolutely) where U_n is any function of n which is $O(n^{-\frac{5}{3}})$.

Now, if we put $$\Gamma(\tfrac{1}{3})/(2^{\frac{2}{3}}3^{\frac{1}{2}}\pi) \equiv A,$$

we have* $$J_{\nu+n}(\nu+n) - A(\nu+n)^{-\frac{1}{3}} = O(n^{-\frac{5}{3}}),$$

and so $\sum_{n=1}^{\infty} a_n \{J_{\nu+n}(\nu+n) - A(\nu+n)^{-\frac{1}{3}}\}$ converges (absolutely). Therefore, since $\sum_{n=1}^{\infty} a_n(\nu+n)^{-\frac{1}{3}}$ is *convergent*, we see at once that $\sum_{n=1}^{\infty} a_n J_{\nu+n}(\nu+n)$ is convergent.

Since $\sum_{n=0}^{\infty} a_n J_{\nu+n}(\nu+n)$ is convergent, and since, whenever $0 \leqslant x \leqslant 1$, the fraction $$\frac{J_{\nu+n}\{(\nu+n)x\}}{J_{\nu+n}(\nu+n)}$$

is a function of n which *does not increase* as n increases (§ 12), it follows from the test of Abel's type for uniformity of convergence that

$$\sum_{n=0}^{\infty} a_n J_{\nu+n}(\nu+n) \frac{J_{\nu+n}\{(\nu+n)x\}}{J_{\nu+n}(\nu+n)}$$

is *uniformly convergent* (and is therefore a continuous function of x) throughout the closed interval $0 \leqslant x \leqslant 1$. And this is the result which had to be established.

[If $\Sigma a_n n^{-\frac{1}{3}}$ does not converge, $\Sigma a_n(\nu+n)^{-\frac{1}{3}}$ cannot converge (for if the latter series converged, the former would also, by Abel's test, since $(\nu+n)^{\frac{1}{3}}n^{-\frac{1}{3}}$ is monotonic and bounded); and then, if $a_n(\nu+n)^{-\frac{1}{3}}$ does not tend to zero, $a_n J_{\nu+n}(\nu+n)$ cannot tend to zero, so that $\Sigma a_n J_{\nu+n}(\nu+n)$ does not converge; while, if $a_n(\nu+n)^{-\frac{1}{3}}$ does tend to zero,

$$a_n\{J_{\nu+n}(\nu+n) - A(\nu+n)^{-\frac{1}{3}}\} = o(n^{-\frac{2}{3}}),$$

so that $\Sigma a_n\{J_{\nu+n}(\nu+n) - A(\nu+n)^{-\frac{1}{3}}\}$ does converge, while $A\Sigma a_n(\nu+n)^{-\frac{1}{3}}$ does not. By adding the last two series we see that $\Sigma a_n J_{\nu+n}(\nu+n)$ cannot converge if $\Sigma a_n n^{-\frac{1}{3}}$ does not do so; this proves the statement made earlier in the section.]

18. *The convergence of Kapteyn series when* $x \geqslant 1$.

The theorem of § 17 has no very simple analogy for the range $x \geqslant 1$. There is an evident but much less delicate theorem that if $\sum_{n=1}^{\infty} a_n n^{-\frac{1}{3}}$

* See § 10, footnote.

converges *absolutely*, then the series $\sum_{n=0}^{\infty} a_n J_{\nu+n}\{(\nu+n)x\}$ converges absolutely and uniformly in *any* range of real values of x. To prove this result, we prove (as in § 17) that $\Sigma a_n J_{\nu+n}(\nu+n)$ converges absolutely, and then take account of the fact (§§ 14, 15) that

$$|J_{\nu+n}\{(\nu+n)x\}| < J_{\nu+n}(\nu+n) \times 1{\cdot}5868\,;$$

the uniformity and absoluteness of the convergence then follow from the test of Weierstrass.

To discuss the uniformity of the convergence when $x \geqslant 1$, we should have to establish an inequality somewhat resembling that of § 12, and the integrals of Debye's type are of a much more complicated form when $x \geqslant 1$ than when $x \leqslant 1$. Moreover, in view of the oscillatory character of $J_{\nu+n}\{(\nu+n)x\}$, which is exhibited by Debye's integrals when $x > 1$, it seems clear that any discussion of the best theorem concerning the uniformity of convergence of a Kapteyn series will involve the consideration of the uniformity of convergence of certain trigonometrical series. It consequently seems desirable to reserve this investigation until some future occasion.

ON NON-ABSOLUTELY CONVERGENT, NOT NECESSARILY CONTINUOUS, INTEGRALS

By W. H. Young.

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1. That absolutely convergent integrals taken with respect to discontinuous functions of bounded variation are in general discontinuous functions, namely at the points of discontinuity of the function $g(x)$ with respect to which the integration takes place, is evident. It is clear also that the extension of these integrals further, so as to embrace the case where the convergence is not absolute, will not have the effect of removing the discontinuity of the integral, even if we adopt the postulates formulated by Harnack, and afterwards investigated by E. H. Moore, supposing, that is, that these conditions are modified so as to take account of not only the work of Lebesgue, but also the results of the theory of integration with respect to discontinuous functions.

It will be sufficient to suppose that there is only a single Harnack point of $f(x)$ with respect to $g(x)$ to make this clear. If this point be a, we should naturally define the integral $\int_0^a f(x)\,dg(x)$ from 0 to a to be the unique limit, supposed to exist, of the integral from 0 to a', as a' moves up to a, together with the integral of the function which is $f(a)$ at the point a and zero elsewhere, taken with respect to $g(x)$. If $g(x)$ is discontinuous at a, there will be a discontinuity of the integral, so defined, at this point.

As soon, however, as we abandon the requisite of continuity, it no longer seems inevitable that an integral should be defined only as the unique limit of an integral over a smaller interval. Thus in the example just given, if any convenient law suggested itself, it would be open to us to consider as an integral the limit of $\int_0^{a_n} f(x)\,dg(x)$, as, n approaching infinity, a_n approaches a according to such a convenient law. We should

in this way get discontinuity at a, even when $g(x)$ is continuous there. To the objection that we should only have *an* integral, and not *the* integral, the answer is easy: the question is not as to whether a similar process may not lead to a different result, but as to whether a sufficiently well defined analytical process is being defined, and the equally relevant question as to whether the application of this process can be of service to us.

Take another instance, and consider the definition given by Harnack, in the case in which the Harnack points form a perfect set of content zero. The integral, supposing the Harnack conditions to be satisfied, is then the sum of the series whose single terms are the integrals over the black intervals of this set. If we suppose a less stringent set of conditions than those of Harnack, the case would arise in which this series did not converge absolutely. In this case the sum of the series would depend on the order in which we took the intervals. If then we are able to give a well defined law for the summation which is of service to us mathematically, we are certainly at liberty to define the integral as the sum of the series, obtained in this manner.

An integral, so defined, may be expected not to possess all the properties of an ordinary integral. But this is a phenomenon to which we are accustomed. Even in the extension of the concept of number, certain well defined properties are lost. If it is shown that a property generally regarded as characteristic of an integral is lost in our generalisations, this may be at most a reason for refusing the name "integration" to the process, but not one for refusing to employ it. In the question of name all will turn on whether the advantages in retaining it, such as the importance of the properties which the retention of the name suggests as possessed by the new concept, have sufficient weight compared with the danger of a property which has been sacrificed being erroneously attributed to it.

Thus in the case of the Lebesgue integrals, there seems every reason for using the word "integrable" instead of "summable," although the latter term has been employed by Lebesgue, to express the property of possessing an integral in his sense; and even in the case of the Harnack and the Denjoy integrals it seems almost pedantic to attempt to avoid the term.

I propose in the present communication, confining myself for simplicity to the case in which the function with respect to which we integrate is the variable x itself, to consider certain generalisations of the usual processes in dealing with functions which possess Harnack points, so that a Lebesgue integral does not exist. The justification for these generalisations may be said to be threefold:—

(i) *The laws defining them are well defined.*

(ii) *The integrals so defined possess some of the most important properties of ordinary integrals, in particular the process of integration by parts is allowable, and the Second Theorem of the Mean is valid.*

(iii) *Their consideration throws light on the properties of an important class of discontinuous functions, and gives us a process for the calculation of the integrals of such functions with respect to functions of bounded variation.*

This fact last mentioned enables us theoretically to calculate the Cesàro sum of certain trigonometrical series, *e.g.* those formed by multiplying term-by-term the Fourier series of such a bounded function by the coefficients of the derived series of the Fourier series of a function of bounded variation.

It will be seen that it is precisely the non-continuity of these integrals which renders them serviceable. On the other hand, the important property that the integral of a sum is the sum of the integrals, has to be abandoned in its complete generality, though it is still true if one of the two functions is summable, or has only a finite number of Harnack points, or has a countable set of Harnack points, provided in the latter case the integral is continuous, or satisfies a less stringent condition,* imposed by the Harnack points of the other function. It should be noted however that, in saying the theorem is not true, we only mean that, if $f(x)$ and $g(x)$ possess integrals, that is, are such that our process is applicable, it does not follow that the process is applicable to $f(x)+g(x)$. It is still open to us to use the sum of the integrals of $f(x)$ and $g(x)$ as an integral of $f(x)+g(x)$, and to investigate the properties of this sum-function by means of its two components.

2. While the interest of the present communication is then partly philosophical, it appears to me to have some bearing on fundamental problems of functions of a real variable. But further the generalisation of the Theorem of Integration by Parts and the Second Theorem of the Mean obtained, include theorems with regard to Denjoy integration, and even special cases of this process, which, as far as I know, have not yet been proved.

From this point of view the part of the paper relating directly to the Theorem of Integration by Parts may be regarded as a proof of this

* § 36 below.

theorem without the use of the Harnack-Moore-Denjoy* condition. The Second Theorem of the Mean is shown to be an immediate consequence of the Theorem of Integration by Parts, in the form which we have given to it. It should be noticed, however, that the Theorem of the Mean has very much diminished in importance since the introduction of integrals with respect to functions of bounded variation. By the use of our form of the Theorem of Integration by Parts, we can avoid the use of the Theorem of the Mean. Moreover we gain very considerably by doing so, not only in precision, but in brevity, and simplicity of treatment.

We are thus able to treat the case of a function which is partly an integral and partly a function of bounded variation, as a whole, instead of considering it separately in different intervals as, for instance, I was obliged to do in my paper on "Multiple Fourier Series."†

3. To readers who are acquainted with Denjoy's two brief Notes‡ in the *Comptes Rendus* (1912), the present paper will, I hope, be of use. It contains a formal proof of the fact that the Harnack-Moore-Denjoy condition is necessary in order that in all cases the integral of the sum should be the sum of the integrals, and also that this is sufficient. On the other hand, it shows that, if continuity of the integral alone is required, this condition is unnecessary, a much less stringent condition being sufficient. On the other hand, if the integral is not continuous, it may be finite everywhere, and yet have infinite discontinuities. It has the property in any case of assuming in any interval all values between its upper and lower bounds in that interval, being, in point of fact, the limit of a continuous function, whose value everywhere is one of the limits for approach from the right as well as one of the limits for approach from the left.§

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* E. H. Moore, "Concerning Harnack's Theory of Improper Definite Integrals," 1901, *Trans. Amer. Math. Soc.*, Vol. 2, pp. 296–330. The condition in its primitive form is given on p. 324, and in Hobson's *Theory of Functions of a Real Variable*, p. 384. See below, § 37.

† *Proc. London Math. Soc.*, Ser. 2, Vol. 11, pp. 133–184.

‡ Arnaud Denjoy, "Une extension de l'intégrale de M. Lebesgue," April 1st, 1912; "Calcul de la primitive de la fonction dérivée la plus générale," April 22nd, 1912, *Comptes Rendus*, Vol. 145.

§ I may perhaps be permitted to conclude these introductory remarks by referring to the fact that the considerations here exposed took their rise in ideas which I conceived as early as 1910. Indeed the footnote on p. 425 of my paper "On the Condition that a Trigonometrical Series should have the Fourier Form" (*Proc. London Math. Soc.*, Ser. 2, Vol. 9), originally contained a reference to the possibility of extensions of the nature of those employed subsequently by Denjoy with such success. It did not seem justifiable to retain this reference, as there was little prospect at the time of my being able to develop the theory. I ought,

In the same connection it should be remarked that Moore, and the writers who have followed him, previous to Denjoy, have made the mistake, as it would appear to be, of requiring the Harnack condition to apply, not only to the black intervals of the perfect nucleus, but also to those of the complete closed Harnack set. It is in this way that Moore was led to conclude that the sum of the integrals of $f(x)$ and $g(x)$ is not the integral of $f(x)+g(x)$. Moreover in the proof of the Theorem of Integration by Parts for Harnack-Lebesgue Integration given by Hobson,* this condition has been assumed to hold in this manner, and has been utilised, whereas, as mentioned in the preceding article, in the present paper this condition is not only not utilised in dealing with the countable adherences but is not required at all, even in connection with the Harnack nucleus.

Whereas, as I have shown, considerations connected with the Theory of Sets of Points may, in dealing with absolutely convergent integrals, be relegated with great advantage to the later chapters of the theory, their almost immediate consideration in the theory of non-absolutely convergent integrals is unavoidable. The same remark is not quite true in regard of transfinite numbers. A knowledge of them is an advantage, but it is not indispensable.

4. The success of the method to be employed depends on the systematic use of certain auxiliary theorems. We proceed to give these first of all.

LEMMA 1.—*If $u(x)$ is a bounded function of x, and $g(x)$ a monotone*

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too, to add that in one essential point Denjoy's theory goes beyond what I had imagined: the introduction by him, not only of Harnack points, but of points which we might with propriety call Denjoy points, namely Harnack points which are such that no interval, however small, can be drawn to enclose one of them, such that the integrals over the black intervals of the Harnack nucleus contained in it obey the Harnack-Moore condition. To this would correspond in the present theory the consideration of the points (the J-points of the present paper) for which this is true of the absolute convergence of the integrals themselves over these black intervals.

The removal of the Harnack-Moore condition in its *sine qua non* form, was one of the subjects of investigation that I had in mind at the time above referred to. But it was not till the early summer of last year that I could find the time to go into the matter. Apart from this I had for some time the excuse for not investigating these matters that Denjoy's two Notes had not, and has not, yet been followed by the detailed development of his theory.

* E. W. Hobson, "The Second Mean Value Theorem of the Integral Calculus," 1908, *Proc. London Math. Soc.*, Ser. 2, Vol. 7, pp. 14–23.

increasing function,

$$\int_{b-0}^{b} u(x)\,dg(x) = u(b)\,[g(b)-g(b-0)],$$

and

$$\int_{a}^{a+0} u(x)\,dg(x) = u(a)\,[g(a+0)-g(a)].$$

Let c be a point near b on the left, and b_n a point between c and b, which, as $n \to \infty$, moves up to b as limiting point.

Denoting by U and L the upper and lower bounds of $u(x)$ in the closed interval (c, b), let us define two functions as follows:

$$v_n(x) = u(b), \quad \text{when} \quad b_n < x \leqslant b, \quad \text{and} \quad v(x) = U \quad \text{elsewhere};$$

$$w_n(x) = u(b), \quad \text{when} \quad b_n < x \leqslant b, \quad \text{and} \quad w(x) = L \quad \text{elsewhere.}$$

Then

$$\operatorname*{Lt}_{n\to\infty} w_n(x) \leqslant u(x) \leqslant \operatorname*{Lt}_{n\to\infty} v_n(x),$$

$$\int_c^b \operatorname*{Lt}_{n\to\infty} w_n(x)\,dg(x) \leqslant \int_c^b u(x)\,dg(x) \leqslant \int_c^b \operatorname*{Lt}_{n\to\infty} v_n(x)\,dg(x).$$

But, since the $v_n(x)$ and $w_n(x)$ sequences are bounded, the integral of the limit is the limit of the integral in each case. Therefore, expressing the integrals and taking the limits, we have

$$\operatorname*{Lt}_{n\to\infty} \{u(b)\,[g(b)-g(b_n)]+L\,[g(b_n)-g(c)]\}$$
$$\leqslant \int_c^b u(x)\,dg(x) \leqslant \operatorname*{Lt}_{n\to\infty} \{u(b)\,[g(b)-g(b_n)]+U\,[g(b_n)-g(c)]\}.$$

This is the same as the following:

$$u(b)\,[g(b)-g(b-0)]+L\,[g(b-0)-g(c)]$$
$$\leqslant \int_c^b u(x)\,dg(x) \leqslant u(b)\,[g(b)-g(b-0)]+U\,[g(b-0)-g(c)].$$

Letting $c \to b$, we get the first of the required equations. Similarly the second may be proved.

5. The two following theorems are required to prove that we may employ the method in Integration by Parts for our Y integrals:—

AUXILIARY THEOREM I.—*If* (a_1, b_1), (a_2, b_2), ... *are intervals, each lying inside the succeeding, and* (a, b) *is the limiting interval, and if* $F(x)$ *is a*

bounded function of x such that $F(a_n) \to F(a)$ and $F(b_n) \to F(b)$, when $n \to \infty$, then

$$\operatorname*{Lt}_{n\to\infty} \left\{ \Big[g(x)\, F(x) \Big]_{a_n}^{b_n} - \int_{a_n}^{b_n} F(x)\, dg(x) \right\} = \Big[g(x)\, F(x) \Big]_a^b - \int_a^b F(x)\, dg(x), \quad (1)$$

where $g(x)$ is a monotone increasing function of x.

This follows immediately from the preceding Lemma, since

$$\int_{a_n}^{b_n} F(x)\, dg(x) = \int_a^b F(x)\, dg(x) - \int_a^{a_n} F(x)\, dg(x) - \int_{b_n}^b F(x)\, dg(x).$$

AUXILIARY THEOREM II.—*If $F(x)$ is a bounded function of x, so defined that—*

(i) *its increments over the black intervals (a_n, b_n) of a certain perfect set form an unconditionally convergent series, and*

(ii) *the sum of these increments corresponding to the black intervals in (x_1, x_2) has $F(x_2) - F(x_1)$ for sum, whenever x_1 and x_2 are points of the perfect set;*

then $F(x)$ is continuous with respect to the perfect set, and

$$\sum_{n=1}^{\infty} \left\{ \Big[g(x)\, F(x) \Big]_{a_n}^{b_n} - \int_{a_n}^{b_n} F(x)\, dg(x) \right\}$$

converges unconditionally, and has for sum

$$\Big[g(x)\, F(x) \Big]_a^b - \int_a^b F(x)\, dg(x),$$

$g(x)$ being a monotone increasing function, and a and b the extreme left- and right-hand points of the set.

First we remark that, since the series of increments is unconditionally convergent, the corresponding series of moduli converges.

Now, if x is a point of the perfect set, and $x+h$ another point of the set, as near to x as we please, we have, by hypothesis,

$F(x+h) - F(x)$ = sum of increments of F over the black intervals in

$$(x,\ x+h)\ ;$$

therefore, if h is chosen less than the length of the least of the first m of the black intervals, when arranged in any chosen order,

$$| F(x+h) - F(x) | \leqslant \sum_{n=m+1}^{\infty} | F(b_n) - F(a_n) |\ ,$$

which, since the series of moduli converges, is as small as please, when m is chosen sufficiently large. This proves the continuity of $F(x)$ with respect to the perfect set.

To prove the second result required, let us find m so that the sum of the increments, taken positively, of $F(x)$ over the first m intervals, when arranged in any chosen order, differs from the whole sum by less than e/M, where M is the upper bound of $g(x)$, supposed, for convenience, positive and monotone increasing.

Then denoting the complementary intervals in order from left to right by (c_1, k_1), (c_2, k_2), ..., (c_s, k_s), the end-points c_r, k_r, being in their entirety identical with the points a_n, b_n $(n \leqslant m)$, together possibly with a and b, are points of the perfect set. Hence the sum of the increments of F over the complementary intervals is the sum of the increments over all but the first m black intervals; the former sum, when its terms are all taken positive, is therefore less than e/M. Therefore also

$$-e \leqslant \sum_{r=1}^{s} g(k_r)\{F(k_r)-F(c_r)\} \leqslant e.$$

That is

$$-e \leqslant \sum_{r=1}^{s} [g(k_r)F(k_r)-g(c_r)F(c_r)] - \sum_{r=1}^{s} F(c_r)[g(k_r)-g(c_r)] \leqslant e, \quad (1)$$

or, what is the same thing, since the intervals (c_r, k_r) are complementary to the intervals (a_n, b_n),

$$-e \leqslant g(b)F(b)-g(a)F(a) - \sum_{r=1}^{m} [g(b_r)F(b_r)-g(a_r)F(a_r)] - \sum_{r=1}^{s} F(c_r)[g(k_r)-g(c_r)] \leqslant e. \quad (2)$$

But the second of the two summations which occur in (2) is

$$\sum_{r=1}^{s} \int_{c_r}^{k_r} F(c_r)\, dg(x),$$

using which, and the identity

$$\int_a^b F(x)\, dg(x) = \sum_{n=1}^{m} \int_{a_n}^{b_n} F(x)\, dg(x) + \sum_{r=1}^{s} \int_{c_r}^{k_r} F(x)\, dg(x), \quad (3)$$

we have, by (2),

$$-e \leqslant \left\{ \Big[g(x)F(x) \Big]_a^b - \int_a^b F(x)\, dg(x) \right\} - \sum_{n=1}^{m} \left\{ \Big[g(x)F(x) \Big]_{a_n}^{b_n} - \int_{a_n}^{b_n} F(x)\, dg(x) \right\} - \sum_{r=1}^{s} \int_{c_r}^{k_r} [F(c_r)-F(x)]\, dg(x) \leqslant e. \quad (4)$$

Now $$\sum_{r=1}^{s} \int_{c_r}^{k_r} \{F(c_r) - F(x)\}\, dg(x) = \int_a^b h_m(x)\, dg(x),$$

where $h_m(x) = F(c_r) - F(x)$ in the interval (c_r, k_r) for each of the complementary intervals, and in the chosen m black intervals $h_m(x) = 0$. Thus, if x is a point not belonging to the perfect set, $h_m(x)$ will certainly have the unique limit zero as $m \to \infty$; for at some stage x will be internal to a chosen black interval.

On the other hand, if x is a point of the perfect set, it will at every stage be internal to one of the complementary intervals (c_r, k_r), and as $m \to \infty$, the end-point c_r of this interval will move up to x as limit along the perfect set.

Since therefore, as we saw, $F(x)$ is continuous with respect to the perfect set, $h_m(x) = F(c_r) - F(x)$ will again have zero as limit. Thus $h_m(x) \to 0$ at every point of (a, b). Since then $\int_a^b h_m(x)\, dg(x)$ is a bounded function of x, it also approaches zero as limit, when $m \to \infty$, since term-by-term integration of the h_m-sequence is allowable.

Thus, letting $m \to \infty$ in (4), we obtain

$$-e \leqslant \left\{ \Big[g(x) F(x) \Big]_a^b - \int_a^b F(x)\, dg(x) \right\}$$

$$- \sum_{r=1}^{\infty} \left\{ \Big[g(x) F(x) \Big]_{a_r}^{b_r} - \int_{a_r}^{b_r} F(x)\, dg(x) \right\} \leqslant e.$$

Since e is as small a positive quantity as we please, this proves that the central member of this inequality has the value zero. But this remains true however we rearrange the series of black intervals; therefore the series involved in that second member is unconditionally convergent, and has the required expression for sum.

This proves the theorem when $g(x)$ is positive, whence, as is at once evident, the truth of the theorem in the general case at once follows.

6. We next have two theorems which enable us to prove that our Y-integrals are functions of their upper limit of integration, which assume all values between their upper and lower bounds.

AUXILIARY THEOREM III.—*If $F(x)$ is a function, whose values at a and b are among the limits of the values of $F(x)$ inside the interval (a, b) in the neighbourhood respectively of a and b, and if in certain closed intervals (α_n, β_n) $(n = 1, 2, \ldots)$, inside the completely open interval (a, b),*

and expanding to (a, b) *as limiting interval,* $F(x)$ *assumes all values between its upper and lower bounds in* (α_n, β_n), *then* $F(x)$ *also assumes all values between its upper bounds in the closed interval* (a, b).

It is evident that $F(x)$ assumes all values between its values at any two points inside the completely open interval (a, b); it is therefore only necessary to show that it assumes all values between its values at any one end-point, and its value at one internal point.

Let then $a_1, a_2, \ldots$ be such a sequence of points with a as limit that $F(a) = \underset{n\to\infty}{\mathrm{Lt}}\, F(a_n)$, and let X be any internal point of (a, b). Then if k is any value between $F(a)$ and $F(X)$ (not inclusive), we can find a_n so that k lies between $F(a_n)$ and $F(X)$. Therefore the value k is assumed by $F(x)$ at some point of the interval (a_n, X).

Hence the theorem at once follows.

AUXILIARY THEOREM IV. — *If* $F(x)$ *is continuous with respect to a perfect set, and in each closed black interval of the perfect set assumes all values between its upper and lower bounds in that black interval, then* $F(x)$ *assumes all values between its upper and lower bounds in the whole interval considered.*

For, since $F(x)$ is continuous with respect to the perfect set, it assumes all values between any two of its values on the perfect set. Also, if x is a point of the perfect set, and y a point not belonging to that set, we only have to interpolate that end-point of the black interval in which y lies, and which lies between x and y, to see that $F(x)$ assumes all values in (x, y) between $F(x)$ and $F(y)$. Hence the theorem follows.

7. The following lemma and theorem enable us to prove the Theorem of the Mean for our Y-integrals very simply.

LEMMA.—*If* $F(x)$ *is a bounded function which assumes all values between its upper and lower bounds in any interval, and is integrable with respect to a monotone increasing function* $g(x)$, *then*

$$\int_a^b F(x)\, dg(x) = F(X)\,[g(b)-g(a)], \qquad (1)$$

where X *is some point of* (a, b).

In fact, if U and L denote the upper and lower bounds of $F(x)$ in the closed interval (a, b),

$$L\,[g(b)-g(a)] = \int_a^b F(x)\, dg(x) = U\,[g(b)-g(a)]. \qquad (2)$$

The sign of equality can evidently only be taken in this relation if $F(x)$ is constant throughout (a, b), exception being made possibly of a set of content zero. In this case the point X may be taken to be any point of (a, b) not belonging to the exceptional set of content zero. If neither of the two possible signs of equality hold in (2), we must have

$$\int_a^b F(x)\, dg(x) = M[g(b)-g(a)],$$

where M lies between L and U, and is therefore a value assumed by F at some point X of (a, b). This proves the lemma.

AUXILIARY THEOREM V.—*If $F(x)$ is a bounded function which assumes all values between its upper and lower bounds in any interval, and is such that $F(x)$ is one of the limits of $F(x-h)$ and of $F(x+h)$ $(0<h)$ as $h\to 0$, and $f(x)$ is a function connected with $F(x)$, by the formula for Integration by Parts, namely,*

$$\int_a^b f(x)\, g(x)\, dx = \Big[F(x)\, g(x)\Big]_a^b - \int^b F(x)\, dg(x),$$

$g(x)$ being a monotone increasing (decreasing) function, then

$$\int_a^b f(x)\, g(x)\, dx = g(a)\,[F(X)-F(a)]+g(b)\,[F(b)-F(X)], \tag{1}$$

where X is some point of (a, b); and if $g(x)$ is positive and monotone decreasing,

$$\int_a^b f(x)\, g(x)\, dx = g(a)\,[F(X)-F(a)]. \tag{2}$$

The equation (1) results at once from the formula for Integration by Parts, using the preceding lemma. To prove (2) we write (1) in the form

$$\int_a^b f(x)\, g(x)\, dx = g(a)\left\{[F(X)-F(a)]+\frac{g(b)}{g(a)}\,[F(b)-F(X)]\right\},$$

and remark that, since $g(x)$ is positive and monotone decreasing,

$$\frac{g(b)}{g(a)}\,[F(b)-F(X)]$$

lies between zero and $F(b)-F(X)$. Also

$$F(x)-F(X)$$

assumes the value $F(b)-F(X)$ at the point b, and, since $F(x)$ approaches $F(X)$ as limit, by hypothesis, when x approaches X along a suitable sequence of points of the interval (X, b), it is clear that $F(x)-F(X)$ assumes values as near zero as we please, and therefore assumes all values between zero and $F(b)-F(X)$. In particular, therefore, it assumes the value $\frac{g(b)}{g(a)}[F(b)-F(X)]$ at some point X' of (X, b). Hence

$$\int_a^b f(x)\,g(x)\,dx = g(a)\,[F(X)-F(a)+F(X')-F(X)]$$

$$= g(a)\,[F(X')-F(a)]\,.$$

which proves the theorem.

8. Let $f(x)$ be the function to be integrated.

First suppose $f(x)$ to have an absolutely convergent integral (Lebesgue integral) in the closed interval (a, b). This is then a special case of a Y-integral.

Next suppose that this is not the case; then the closed interval (a, b) necessarily contains *Harnack points*, namely, points such that no closed sub-interval, over which $f(x)$ is absolutely integrable, can be drawn, however small, containing such a point. These Harnack points necessarily form a closed set, say H, *the Harnack set*. This closed set H may be countable, but will in general consist of a countable set and a perfect nucleus, and we shall proceed systematically to classify the integrals we are going to define, according to the constitution of the set H, analysed, as Cantor did, into its successive adherences and its nucleus. In the present communication I do not go into all the details of this process, but I call attention at the proper places to the new features of the present definition.

9. *When the Harnack points are finite in number, say m,* the definition of the Y-integral is again familiar; it is that which has been hitherto adopted for the simplest case of a Harnack-Lebesgue integral. The formula is

$$\int_a^b f(x)\,dx = \sum_{n=1}^{m} \operatorname*{Lt}_{(a_n'\to a_n;\ b_n'\to b_n)} \int_{a_n'}^{b_n'} f(x)\,dx. \qquad (1)$$

The condition for the existence of this integral being that the Lebesgue

integral $\int_{a_n'}^{b_n'} f(x)\,dx$, which necessarily exists when a_n' and b_n' lie in the same compartment bounded by Harnack points, should have a unique limit when $a_n' \to a_n$ and $b_n' \to b_n$ in any manner.

10. If we replace the upper limit b of integration by a variable x, remaining in the interval (a, b) in which the requisite conditions hold, we obtain a function

$$F(x) = \int_a^x f(x)\,dx + \text{const.},$$

involving an arbitrary constant. This is the indefinite integral.

$F(x)$ is, as is the case of a Lebesgue integral, a continuous function of x, and it is easily proved to have the following four fundamental properties :—

(κ) $\int f(x)\,g(x)\,dx$ *exists, when* $g(x)$ *is a monotone increasing (decreasing) function ;*

(λ) *the Theorem of Integration by Parts holds, namely :—*

$$\int_a^x f(x)\,g(x)\,dx = \Big[F(x)\,g(x)\Big]_a^x - \int_a^x F(x)\,dg(x)\,;$$

(μ) $F(x)$ *assumes all values between its upper and lower bounds in any closed interval* $(a \leqslant x \leqslant b)$, *and* $F(x)$ *is one of the limits of* $F(x+h)$ *and* $F(x-h)$, $(0 < h)$, *as* $h \to 0$;

(ν) *the Second Theorem of the Mean holds in its two familiar forms, namely,*

$$\int_a^b f(x)\,g(x)\,dx = g(a)\int_a^X f(x)\,dx + g(b)\int_X^b f(x)\,dx,$$

where $g(x)$ *is a monotone increasing (decreasing) function, and* x *is some point of* (a, b), *and*

$$\int_a^b f(x)\,g(x)\,dx = g(a)\int_a^X f(x)\,dx,$$

when $g(x)$ *is monotone decreasing.*

Indeed (μ) is well known to be true for any continuous function, and, by our Auxiliary Theorem V, the last property (ν) holds for any function having the first three properties. Also it is evident that the Harnack

points of the function $f(x)\,g(x)$ are included among those of $f(x)$, so that (κ) will hold, provided the condition as to the uniqueness of the limit of $\int_{a'_n}^{b'_n} f(x)\,g(x)\,dx$ is satisfied under the same conditions as that of $\int_{a'_n}^{b'_n} f(x)\,dx$.

Since the Theorem of Integration by Parts holds for Lebesgue integrals,

$$\int_{a'_n}^{b'_n} f(x)\,g(x)\,dx = \Big[F(x)\,g(x)\Big]_{a'_n}^{b'_n} - \int_{a'_n}^{b'_n} F(x)\,dg(x).$$

Now, by our assumption as to the uniqueness of the limit of $\int_{a'_n}^{b'_n} f(x)\,dx$, the conditions of our Auxiliary Theorem (1) are satisfied. Therefore also $\int_{a'_n}^{b'_n} f(x)\,g(x)\,dx$ has a unique limit, which proves (κ), and this limit is expressed by the right-hand side of the equation for Integration by Parts, which also proves the property (λ). *Thus our Y-integral has, as stated, the four fundamental properties.*

11. *Next suppose the Harnack set to be such that its first derived set H' consists of a finite number m of points.*

Now (a_n, b_n) will denote any one of the compartments determined by the m points, and a'_n, b'_n will have the same meaning as before. Therefore, in (a'_n, b'_n), there is only a finite number of Harnack points; hence $f(x)$ has a Y-integral of the type already discussed in (a'_n, b'_n), supposing the requisite condition satisfied.

In may be that in (a_n, b_n) itself there is only a finite number of Harnack points, in which case the Y-integral of $f(x)$ over (a_n, b_n) is already defined.

Again, it may be that one or other of the end-points a_n or b_n is not a limiting point of the Harnack points, so that that point may be taken to replace the variable limit of integration having it for limiting point.

In general, however, a_n and b_n are limiting points of Harnack points in the interval between them, and instead of making the same assumption as before, we may make the less stringent assumption that, *when the variable points a'_n and b'_n move along the Harnack set to their limiting points a_n and b_n, the integral $\int_{a'_n}^{b'_n}$ has a unique limit.*

We have here departed from tradition. In the definition of a Harnack-Lebesgue integral, the uniqueness of the limit for all modes of approach

to the limiting point has always been hypothecated. But our assumption is sufficient to give us a perfectly determinate definition of the integral of $f(x)$—including the corresponding Harnack-Lebesgue integrals as a special case—and to show that it will still have the four fundamental properties.

The definition of the Y-integral of $f(x)$ will in this case be given by the following formula :—

$$\int_a^b f(x)\,dx = \sum_{n=1}^{m} {}^{*}\mathrm{Lt}_{(a_n' \to a_n\,;\ b_n' \to b)} \int_{a_n'}^{b_n'} f(x)\,dx, \tag{2}$$

where the asterisk before the limit denotes that the approach is in accordance with the analysis of the Harnack set in the corresponding interval, as just explained, the approach being along isolated points of the Harnack set, unless no such mode of approach exists. In this latter case the variable point (e.g., a_n') may be at once replaced by its limiting point (here a_n).

The integrals on the right in this formula are Y-integrals of the Harnack-Lebesgue type, with a finite number of Harnack points.

12. This is the definite Y-integral, and leads as before to the indefinite Y-integral

$$F(x) = \int f(x)\,dx.$$

In any one of the intervals (a_n', b_n'), $F(x)$ possesses the four properties (κ), (λ), (μ), and (ν), by § 7.

Hence, by the definition already given of a Y-integral, $\int_a^b f(x)\,g(x)\,dx$ will exist if we show that $\int_{a_n'}^{b_n'} f(x)\,g(x)\,dx$, which we know exists, has a unique limit under the same conditions as $\int_{a_n'}^{b_n'} f(x)\,dx$, for each value of n. But this is an immediate consequence of the Auxiliary Theorem 1, provided $F(x)$ is a bounded function of x. This disposes of the properties (κ) and (λ).

$F(x)$ is not necessarily any longer a continuous function, but, by the definition, it is still one of the limits of $F(x \pm h)$, as $h \to 0$. Moreover, by the Auxiliary Theorem III, $F(x)$ assumes in each of the closed intervals (a_n, b_n) every value between two of its values, since this is true in each of the intervals (a_n', b_n'); hence $F(x)$ also assumes all values between its upper and lower bounds in (a, b). Thus $F(x)$ has the property (μ). Therefore, using the Auxiliary Theorem V, $F(x)$, if a bounded function of x, possesses the four properties (κ), (λ), (μ), and (ν).

13. The preceding articles suffice, together with the principle of ordinary induction, to explain the definition of a Y-integral, and to show that an indefinite Y-integral $F(x)$ has the four properties (κ), (λ), (μ), and (ν), provided it is a bounded function of x, whenever one of the derived sets of the Harnack set consists of a finite number of points only. The only point to emphasize is that, the condition for the existence of the Y-integral in an interval having one of the Harnack points for end-point, only involves the existence of a unique limit for the integral at points in the neighbourhood, *when those points belong to the preceding derived set;* only in those cases in which, in the interval, in question, the Harnack points do not approach the end-point under consideration, do we need the limit to be unique for all possible modes of approach. On each side of each point the appropriate condition is completely determined by the analysis of the Harnack set.

14. No fresh point is introduced into the discussion of the Y-integral in the case when all the derived sets exist, and the first deduced set consists of m points; indeed in this case (a_n, b_n) denoting any one of the compartments so determined, and (a'_n, b'_n) any interval inside this compartment, the Y-integral in (a'_n, b'_n) is defined by our previous conventions, supposing the required conditions to be satisfied. The convention we make as to the unicity of the limit of $\int_{a'_n}^{b'_n} f(x)dx$ is again indicated by the analysis of the Harnack set.

If in this compartment (a_n, b_n), the points a_n and b_n are not limiting points of all the derived sets, the limiting process is superfluous, a'_n may be at once replaced by a_n and b'_n by b_n. In the contrary case we suppose that $\int_{a'_n}^{b'_n} f(x)dx$ has a unique limit, when a'_n and b'_n move up to a_n and b_n respectively along Harnack points of which one at least—and therefore an infinite number—belongs to each of the derived sets.

The integral is again given by the formula (2), interpreted in accordance with the contents of the present article. Our previous arguments then show that the indefinite integral $F(x)$ still possesses the four fundamental properties.

15. From this point on, no new principle is introduced, as long as we are dealing with a countable set of Harnack points. By Cantor induction we pass from case to case. The formula (2) still holds, (a_n, b_n) denoting always one of the compartments into which (a, b) is divided by the m points

which constitute the last of the derived and deduced sets. By the well known theory of closed countable sets, such a final set always exists; if it is the derived set of another of the derived and deduced sets, the starred limit denotes that the variable point moves along the penultimate set; if this is not the case, the variable point is supposed to move along Harnack points, one at least from each of the sets anterior to the final set, these sets being, by the theory, known to be countable in their entirety.

Thus by a finite or countably infinite set of conventions, we define the Y-integral of any function $f(x)$, having in (a, b) only a countable set of Harnack points.

16. We pass now to the general case, when the Harnack points form a non-countable closed set. By the Theorem of Cantor-Bendixson, we analyse the Harnack set into a countable set and its perfect nucleus. In any one of the closed black intervals (a_n, b_n) of the Harnack nucleus H_N, the Y-integral is defined by our conventions already made, since the Harnack points in (a_n, b_n) are countable. The new conventions which we make are two in number:—

(1) $\sum_{n=1}^{\infty} \int_{a_n}^{b_n} f(x)\,dx$ *is an unconditionally convergent series;*

(2) $f(x)$ *is summable over the perfect set* H_N.

We then define the Y-integral by the formula

$$\int_a^b f(x)\,dx = \int_{H_N} f(x)\,dx + \sum_{n=1}^{\infty} \int_{a_n}^{b_n} f(x)\,dx. \tag{3}$$

In particular, if $f(x)$ is zero at all the points of the Harnack nucleus, excepting a sub-set of content zero,

$$\int_a^b f(x)\,dx = \sum_{n=1}^{\infty} \int_{a_n}^{b_n} f(x)\,dx. \tag{3a}$$

17. We can now prove that our Y-integrals still have the four fundamental properties.

Indeed, in each black interval (a_n, b_n) the properties (κ) and (λ) hold. Hence it is sufficient to prove that

$$\sum_{n=1}^{\infty} \left\{ \Big[g(x)\,F(x) \Big]_{a_n}^{b_n} - \int_{a_n}^{b_n} F(x)\,dg(x) \right\}$$

—which is accordingly the same as $\sum_{n=1}^{\infty} \int_{a_n}^{b_n} f(x)\, g(x)\, dx$ and converges unconditionally—to prove the existence of the Y-integral of $f(x)\, g(x)$ from a to b.

Also the property (λ) will follow, if the sum of this unconditionally convergent series is shown to be

$$\Big[g(x)\, F(x)\Big]_a^b - \int_a^b F(x)\, dg(x).$$

Now in the case of the formula $(3a)$, this is an immediate consequence of the Auxiliary Theorem II. Hence, in the general case, formula (3) may be written

$$F(x) = \int_a^x f(x)\, dx = \int_a^x f_1(x)\, dx + \int_a^x f_2(x)\, dx = F_1(x) + F_2(x), \qquad (4)$$

where the first integral is a Lebesgue integral, and the second, being a Y-integral of the type $(3a)$, is such that

$$\int_a^b f_2(x)\, dg(x) = \Big[g(x)\, F_2(x)\Big]_a^b - \int_a^b F_2(x)\, dg(x).$$

Since, therefore, by the theory of Lebesgue integrals, a similar equation holds for $f_1(x)$, by addition we get a similar equation for $f(x)$. Thus in the general case of formula (3), our Y-integral has again the properties (κ) and (λ).

18. *It is clear that $F(x)$, defined by* (3), *is continuous with respect to the Harnack nucleus;* for, by the Auxiliary Theorem II, $F_2(x)$ is so, while $F_1(x)$ is a continuous function of x; hence, by (4), $F(x)$ is so. Thus, by the Auxiliary Theorem IV, $F(x)$ assumes all values between its upper and lower bounds. Thus our Y-integrals still have the property (μ).

Hence also, by Theorem V, they have all the four fundamental properties (κ), (λ), (μ), and (ν).

19. Next suppose that $f(x)$ is not summable over the Harnack nucleus. Then we analyse this latter set precisely as hitherto we have analysed the fundamental interval (a, b). Supposing intervals to exist in which $f(x)$ is summable over the Harnack nucleus, the conventions already made suffice to determine the Y-integral of $f(x)$ in these, or to convince us that a Y-integral does not exist. We direct our attention then primarily to the remaining points, which are such that $f(x)$ is not summable over the part

of the Harnack nucleus in any interval containing any of them, or having one of them for end-point.

Let the set of these points of non-summability be denoted by H_{N-S}, and let the process of passing from the Harnack set H to the set H_{N-S} be called for brevity *the non-summability process* performed on H; it is the same process which, performed on the fundamental segment, leads to the Harnack set. *The set H_{N-S} is again a closed set.*

We notice that the process of non-summability applied to H, or any of its derived and deduced sets, leads to the same set H_{N-S}, since any Harnack point in the neighbourhood of which the Harnack set is countable, or even of zero content, will be eliminated by the non-summability process.

We now apply the processes of non-summability and deduction, as formerly those of derivation and deduction, and obtain after a countable aggregate of closed sets have been passed over, a final set (not the null-set), contained in all the others. To this set we then apply the processes of derivation and deduction and so arrive at a smallest set, which may be called *the Nucleolus of Non-Summability*, and denoted by $H_{N(N-S)}$. It may be a finite set, or a perfect set of content zero, or else, since the processes of derivation and non-summability applied to it must lead to itself, or to the null-set, it must be a perfect set which is throughout of positive content,* and such that, with respect to it, $f(x)$ is summable, or is totally non-summable, that is to say, there is no interval containing a point of the set, such that $f(x)$ is summable with respect to the sub-set in that interval.

For example, we may suppose that the process of non-summability applied to H_{N-S} leads to a new† set, and applied to this leads again to a new set, and so on *ad infinitum*. The process of deduction then leads necessarily to a fresh set, which may be the Harnack nucleolus, and completes the aggregate of sets; this is, of course, one of the simpler types of such aggregates.

20. Let (a_n, b_n) denote any black interval of the set H_{N-S}. Then, in (a_n, b_n), either there is only a countable set of Harnack points, in which case $\int_{a_n}^{b_n} f(x)\,dx$ has been already defined, or else in any closed interval (a_n', b_n') contained in the completely open interval (a_n, b_n), the Harnack set includes none of its points of non-summability, and consists therefore,

* That is, a set such that in every interval containing one of its points, there is a sub-set of the set of positive content.

† *I.e.*, a set not identical with the former.

apart from points not belonging to the Harnack nucleus, of a sub-set of the latter with respect to which $f(x)$ is summable. Thus $\int_{a'_n}^{b'_n} f(x)\,dx$ is determined by our previous conventions. We shall then, as before, assume that $\int_{a'_n}^{b'_n} f(x)\,dx$ approaches a unique limit, when a'_n and b'_n move up to a_n and b_n respectively, along appropriate sequences. These sequences will be any contained in the Harnack nucleus; if no such sequences exist, the moving point may be replaced by its limiting point in the integral whose limit is under consideration.

21. If then the set H_{N-S} consists of m points, the integral may be defined by the formula (2) of § 11,

$$\int_a^b f(x)\,dx = \sum_{n=1}^{m} {}^{*}\mathrm{Lt}_{(a'_n \to a_n\,;\ b'_n \to b_n)} \int_{a'_n}^{b'_n} f(x)\,dx. \qquad (2)$$

The integral under the sign of summation on the right is now a Y-integral of the most general type hitherto defined, and the approach indicated by the asterisk is along the Harnack nucleus, unless no such mode of approach exists, in which case the moving end-point of the range of integration may be replaced at once by its limiting point in the formula.

In this case H_{N-S} is itself the Nucleolus of Non-Summability, and we may specify the intervals (a_n, b_n) in the formula (2) to be the black intervals of this nucleolus. In this form the formula is valid for the more complicated cases which may arise, always supposing the Nucleolus of Non-Summability to consist of m points, and the Y-integrals on the right to have been defined by previous conventions. Thus in particular we have the definition when the repetition any finite number of times of the process of non-summability leads to a finite number of points only, *the asterisk referring in each case to the preceding set*, or when, after an infinite number of such applications, the process of deduction leads to a finite set, *the asterisk then referring to all the sets, without omitting any one.*

22. When, on the other hand, H_{N-S} consists of a perfect set with respect to which $f(x)$ is summable, the definition is given by the formula

$$\int_a^b f(x)\,dx = \int_{H_{N-S}} f(x)\,dx + \sum_{n=1} {}^{*}\mathrm{Lt}_{(a'_n \to a'_n\,;\ b_n \to b_n)} \int_{a'_n}^{b'_n} f(x)\,dx, \qquad (5$$

supposing the series on the right to converge unconditionally.

The asterisk denotes approach along the preceding set, that is H_N, unless no such mode exists, in which case as usual the limiting point may be at once substituted for the point approaching it.

This formula is included in the more general one

$$\int_a^b f(x)\,dx = \int_{H_N(N-S)} f(x)\,dx + \Sigma \underset{(a'_n \to a_n\,;\, b'_n \to b_n}{{}^*\mathrm{Lt}} \int_{a'_n}^{b'_n} f(x)\,dx, \qquad (6)$$

which is valid whenever the Nucleolus of Non-Summability is perfect, supposing the integrals on the right already defined by previous conventions, and the series to converge unconditionally.

Thus by recurrence and Cantor induction we have the definition when the Nucleolus of Non-Summability is the first, second, ... set obtained by the repeated application of the process of non-summability, or any of the subsequent sets of our aggregate, always supposing that the necessary condition of the preceding article is fulfilled, so that we are not baffled by the appearance of a set with respect to which $f(x)$ is totally non-summable, and that at every stage the series obtained converge unconditionally. In all other cases which may arise in this way we have the definition of the Y-integral.

23. We have not yet exhausted all the possible cases in which $f(x)$ has a Y-integral. We have hitherto hypothecated that the series of integrals obtained at the various stages, as described above, should converge unconditionally. In many instances, however, a less stringent hypothesis is adequate.

The first stage at which this condition came into force was marked by our arrival at the Harnack nucleus H_N. In the black intervals of H_N the integrals of $f(x)$ were defined by our former conventions, and we then required the condition of unconditional convergence of these integrals, so as to form the Y-integral, if $f(x)$ was summable over H_N.

We may express the non-fulfilment of *the condition of unconditional convergence, or J-condition*, as I shall for brevity call it, by saying that one at least of the points of the Harnack nucleus is such that no interval, whose end-points are points of the Harnack nucleus, can be drawn surrounding it, in which the J-condition is satisfied, *i.e.* the integrals of $f(x)$ over the black intervals of H_N in it form an unconditionally convergent series. I shall call such a point a J-point of the Harnack nucleus, and the sub-set of all such points the J sub-set of H_N; it is evidently a closed sub-set.

We can now analyse the perfect set H_N by the J-process, and the

process of deduction, precisely as before, the J-process taking the place of the non-summability process. As before, therefore, after a countable set of stages, we are bound to arrive at a final set, *the J-nucleolus*, which is either a finite set, or a perfect set. In the latter case it must either consist entirely of J-points, in which case there is no Y-integral, or it must contain no J-points, since the J-process leads to no fresh set; when this occurs we are able to define the Y-integral, supposing $f(x)$ summable over the Harnack nucleus.

The formulæ are precisely the same as before. Denoting the J subset of H_N by H_{JN}, the integral of $f(x)$ over any interval inside a black interval of H_{JN} is defined by our former conventions, provided the end-points of the interval belong to H_N. Hence, supposing the limit of this integral unique, as it expands to the black interval as limiting interval, the integral of $f(x)$ over the black interval itself is obtained by means of a starred limit. Over H_{JN} our function is summable, since it was supposed summable over the Harnack nucleus; therefore if H_{JN} has itself no J-points, we have the integral defined by the usual formula, namely,

$$\int_a^b f(x)\,dx = \int_{H_{JN}} f(x)\,dx + \Sigma \int_{\alpha_n}^{b_n} f(x)\,dx, \tag{7}$$

where

$$\int_{\alpha_n}^{b_n} f(x)\,dx = \underset{(\alpha_n' \to \alpha_n\,;\; b_n' \to b_n)}{{}^*\mathrm{Lt}} \int_{\alpha_n'}^{b_n'} f(x)\,dx, \tag{8}$$

(α_n, b_n) being any black interval of the J-nucleolus, so that there is a finite number of values of n, if H_{JN} is a finite set, and, if H_{JN} is a perfect set, the series is unconditionally convergent.

It is unnecessary to explain further. These formulæ are perfectly general, and the discussion is precisely the same as before, following our new analysis in place of that formerly used. All the perfect sets which occur, being sub-sets of the Harnack nucleus, are such that $f(x)$ is summable over them, and in their black intervals the integral is already defined by the conventions already made. At the final stage the formulæ for the integral are the formulæ (7) and (8), H_{JN} denoting now the J-nucleolus.

24. If however $f(x)$ is not summable over the Harnack nucleus H_N, the preceding article gives us the integral of $f(x)$ over any interval whose end-points are points of H_N, inside a black interval of the set H_{N-S}. If therefore $f(x)$ is summable over H_{N-S}, we can again define the integral, by the process of the preceding article. If $f(x)$ is not summable over H_{N-S}, we proceed a stage further, and so on. Ultimately, supposing the

requisite condition satisfied at each stage, we arrive at the Nucleolus of Non-Summability, and, if $f(x)$ is summable over it, our whole process comes to an end. *Thus in all cases in a countable number of stages we arrive at the final formulæ of the type* (7) *and* (8), *unless we detect* (i) *an interval in which the starred limit process does not lead to a unique result, or* (ii) *a perfect set with respect to which $f(x)$ is totally non-summable, or* (iii) *a perfect set consisting entirely of J-points.*

In all cases the form of the definition being the same, and the integrals involved being supposed already defined, we have the same arguments as in the earliest stages. Thus the properties (κ), (λ), (μ), and (ν) remains always valid, and are proved by our Auxiliary Theorems.

25. We have seen that the Y-integral $F(x)$ has the properties, united under the heading (μ) in the preceding discussion, of assuming all values between its upper and lower bounds, and being one of the limits of its values on each side. I have proved* elsewhere that either of these properties is the consequence of the other, when the function concerned is the limit of a continuous function, and it naturally suggests itself that $F(x)$ may itself be such a function. This is, in fact, the case, as we may prove in the following manner.

In the first place we see that $F(x)$ is continuous at any point which is not a Harnack point. For in this case there is a nearest Harnack point on the left of the point x, since the Harnack points form a closed set. If X denote the halfway point between x and this Harnack point, we have from the definition

$$F(x)-F(X)=\int_X^x f(x)\,dx,$$

where the integral on the right, being a Lebesgue integral, is continuous, and thus $F(x)$ is continuous at the point x.

Hence if the Harnack points are countable, Baire's condition, which amounts to this, that in every perfect set there must be a point at which $F(x)$ is continuous with respect to the perfect set, is satisfied. Thus in this case certainly $F(x)$ is the limit of a continuous function.

Again, if the Harnack points are not countable, but the conditions as to summability and unconditional convergence with respect to the Harnack nucleus are satisfied, $F(x)$ is, as we saw in § 18, continuous with respect to the Harnack nucleus. If G is any other perfect set, there must be a point P of G not belonging to H_N, therefore also there is an interval d_P

* "A Theorem in the Theory of Functions of a Real Variable," 1907, *Rend. Pal.*, Vol. XXIV, pp. 187–192.

containing no point of H_N, but containing P, and therefore also containing a perfect sub-set of G. In d_P there is therefore certainly a point of G which is not a Harnack point, since the Harnack points in d_P can form at most a countable set. At this point of G the Y-integral is, as we saw, continuous, and therefore *a fortiori* continuous with respect to the perfect set G. Thus again Baire's condition is satisfied, which shows that the Y-integral is the limit of a continuous function.

The general result can be proved by Cantor induction. Our Y-integral determines, by definition, a countable set of closed sets, all sub-sets of the Harnack set, following one another in a certain order, such that there is a last set, contained in all the preceding, which either consists of a finite number of points, or of a perfect set with respect to which $f(x)$ is summable, and therefore $F(x)$ continuous.

If x is any point not belonging to this final set, it is either not a Harnack point at all, and therefore $F(x)$ is continuous there, or it does not belong to some one of the preceding sets, and we may assume, in order to complete our induction, that the theorem being true for that set, $F(x)$ is continuous there, or finally it belongs to all the preceding sets, though not to the final set itself. In this last case, the final set is not obtained by deduction from all the preceding sets, since it does not contain our point. Therefore it is obtained by one of the other processes employed, derivation, non-summability or the J-process, from a definite preceding set, to which, accordingly, our point x belongs, or else $F(x)$ is continuous there. But if x belongs to the preceding set, it follows from the processes mentioned that there is an interval containing x and containing no point of the final set, in which $F(x)$ belongs to a lower type of integral, and therefore is supposed known to be the limit of a continuous function.

Thus if G is any perfect set, not contained in the final set, and therefore containing a point P not belonging to the final set, there will be an interval d_P containing P, and therefore containing a perfect sub-set of G, in which $F(x)$ is the limit of a continuous function. There is accordingly a point of G at which $F(x)$ is continuous with respect to G: thus again Baire's condition is satisfied. This proves that in all cases our Y-integral $F(x)$ is the limit of a continuous function.

26. We have now seen that *the Y-integral function $F(x)$ is the limit of a continuous function, and has, in common with a differential coefficient, the properties of assuming all values between its upper and lower bounds, and being one of its limits on both sides.* When discontinuous, its discontinuities are therefore all of the second kind.

The following is a simple example of a discontinuous Y-integral.

Ex. 1.—Let $f(x) = 1/(b_n - a_n) \quad [a_n < x < \frac{1}{2}(a_n + b_n)]$,

and $f(x) = -1/(b_n - a_n) \quad [\frac{1}{2}(a_n + b_n) \leqslant x < b_n]$,

for each integer n, (a_1, b_1), (a_2, b_2), ... being the black intervals of a perfect set G, of content zero. At the points of G, $f(x)$ may have any values we please.

In this case $\int_{a_n}^{b_n} f(x)\,dx$ is zero for all integers n. Hence

$$\int_a^b f(x)\,dx = 0.$$

The Harnack points are all the points of the perfect set G, but the integral is a Y-integral, not a Harnack-Lebesgue or Denjoy integral. In fact, $F(x)$ is not a continuous function.

At any point of the perfect set, $F(x)$ is zero, and is therefore continuous with respect to that set. But at the middle point of any black interval $F(x) = \frac{1}{2}$; hence at any point of the perfect set which is not an end-point of a black interval

$$F(x) = 0, \quad \bar{F}(x+0) = \tfrac{1}{2}, \quad \underline{F}(x+0) = 0, \quad \bar{F}(x-0) = 0, \quad \underline{F}(x-0) = -\tfrac{1}{2}.$$

At the end-points of the black intervals we have a similar discontinuity on the side away from the black interval. Throughout the closed black intervals $F(x)$ is continuous.

Thus the discontinuities of $F(x)$ coincide with the Harnack points and form the perfect set G.

27. The following is an example of a simple Y-integral which is continuous, without being a Harnack-Lebesgue or Denjoy integral.

Ex. 2.—Let $f(x) = 1/m(b_n - a_n) \quad [a_n < x < \frac{1}{2}(a_n + b_n)]$,

and $f(x) = -1/m(b_n - a_n) \quad [\frac{1}{2}(a_n + b_n) \leqslant x < b_n)]$,

where (a_1, b_1), (a_2, b_2), ... are the black intervals of Cantor's Typical Ternary Set, arranged in order of magnitude, those intervals which are of the same length being arranged so that those to the left precede those on the right, and m denoting the index of the power of $\frac{1}{3}$ which expresses the length of the interval (a_n, b_n); thus for $n = 1$, $m = 1$; for $n = 2$ or 3, $m = 2$; for $n = 4, 5, 6$, or 7, $m = 3$, &c.

Then, as before, $F(x) = 0$ at all points of the perfect set. At the

middle point of (a_n, b_n) we have $F(x) = 1/m$, and between the middle point and either end-point of (a_n, b_n), $F(x)$ is monotone and continuous.

Hence in approaching any point X of the perfect set, we shall obtain the greatest limit by moving from one middle point to another of the black intervals. In doing this the integer n, and therefore also m, will increase indefinitely: therefore $F(x)$ will even in this case approach zero. Thus $F(x)$ is continuous also at the points of the perfect set, and is a continuous function.

But the integral $F(x)$ is not a Harnack-Lebesgue or Denjoy integral.

For, if W_n denote the upper bound of $\int_{a_n}^{x} f(x)\,dx$, when x is any point whatever of (a_n, b_n), Harnack and his followers, E. H. Moore, and others, demand the convergence of the series ΣW_n, and Denjoy demands that, in every interval containing a point of the perfect set, we should be able to find an interval containing a point of the set, such that ΣW_n taken over the black intervals in that interval should converge; this condition Denjoy expresses by saying that the points of the perfect set in the neighbourhood of which ΣW_n diverges must be dense nowhere on the perfect set.

That ΣW_n is a divergent series is evident, which shows that the integral is not a Harnack-Lebesgue integral.

To see that Denjoy's modification of the condition does not hold, we remark that if x is any point of the perfect set, and d_x any interval containing x as internal point, we can determine an integer r, such that the distance of x from the nearer end-point of d_x is greater than 3^{-r}. Hence d_x will certainly contain two consecutive divisions, when the whole segment (0, 1) is divided into 3^{r+1} equal parts. In one of these, by the construction of Cantor's Typical Ternary Set, the sub-set is similar to the whole perfect set, so that the values of m which are connected with the black intervals in d_x will include all integers from and after a certain value. Therefore ΣW_n corresponding to d_x is certainly divergent.

Thus Denjoy's condition is not satisfied, for, in the neighbourhood of *every* point of Cantor's perfect set, ΣW_n diverges.

28. We notice that, in the preceding example, W_n approaches zero as unique limit when $n \to \infty$. This is, in fact, the only condition required to ensure the continuity of a bounded Y-integral, where the integrand is summable with respect to the Harnack nucleus, provided the condition represented by the starred limit (§ 9 *seq.*) is satisfied when we remove the asterisk. The corresponding conditions in the general case are exactly

similar; such conditions must apply at all stages in order that our formulæ may define a continuous function.

It is evidently necessary that the condition represented by the asterisk should hold when we remove the asterisk, and this not merely when the interval considered is one which occurs in our formula, but for any interval whatever. In other words, if $F(x)$ is a continuous Y-integral, and (α, β) any interval,

$$\int_\alpha^\beta f(x)\,dx = \operatorname*{Lt}_{(\alpha' \to \alpha\,;\ \beta' \to \beta)} \int_{\alpha'}^{\beta'} f(x)\,dx,$$

where α' and β' are points inside (α, β), which approach their limiting points in *any manner whatever*.

On the other hand, if G is any perfect set, with respect to which $f(x)$ is summable, and supposing the series of integrals over the black intervals unconditionally convergent, we have, denoting by (a_n, b_n) the typical black interval of G,

$$\int_a^x f(x)\,dx = \left[\int_G f(x)\,dx + \Sigma \int_{a_n}^{b_n} f(x)\,dx\right]_a^x,$$

the symbol $\left[\ldots\right]_a^x$ denoting that we only regard the interval (a, x) and behave as if outside this interval $f(x)$ were zero.

Hence

$$F(x+h) - F(x) = \int_x^{x+h} f(x)\,dx = \left[\int_G f(x)\,dx + \Sigma \int_{a_n}^{b_n} f(x)\,dx\right]_x^{x+h}.$$

The first term inside the bracket on the right is as small as we please, h being sufficiently small, since $\int_G f(x)\,dx$ is a Lebesgue integral. The sum of the integrals over whole intervals (a_n, b_n) is also as small as we please, since the series of integrals over the black intervals is unconditionally convergent. If therefore x and $x+h$ are both points of G, the right-hand side is as small as we please. Thus $F(x)$ is continuous with respect to the perfect set G.

If, however, one or both of the points x and $x+h$ are not points of G, there will be one or two terms left over. Supposing $x+h$ alone not to belong to G, and a_n the left-hand end-point of the black interval in which $x+h$ lies, the term

$$\int_{a_n}^{x+h} f(x)\,dx$$

will remain over if $h > 0$; and similarly, if $h < 0$, an integral will re-

main over whose limits of integration lie in the same black interval, the length of which is as small as we please when h is sufficiently small.

Denoting by W_n, as in the preceding article, the upper bound of the absolute value of $\int_{a'_n}^{b'_n} f(x)\,dx$, when a'_n and b'_n lie in (a_n, b_n), let us suppose that, for an infinite series of integers n, W_n is greater than e.

Let x be a limiting point of the corresponding intervals (a_n, b_n); then x is a point of G and, however small H may be, there will be one of these intervals (a_n, b_n) in the interval $(x, x+H)$, for positive or negative H. Hence in such an interval (a_n, b_n) there will be two points $a'_n = x+h_1$ and $b'_n = x+h_2$, for which the terms left over, above referred to, differ by at least e. Approaching x along the sequence a'_n we must therefore get a limit differing by at least e from any limit obtainable along the sequence b'_n. Thus $F(x+h)-F(x)$ could not have in both cases the unique limit zero. This shows that at such a point x the integral $F(x)$ is discontinuous.

Thus we see that, if $F(x)$ is continuous, W_n must have the unique limit zero, for every perfect set over which $f(x)$ is summable, and with respect to whose black intervals the condition of absolute convergence is satisfied.

29. Conversely this condition, with the former one (the suppression of the asterisk), are sufficient to ensure the continuity of $F(x)$, supposed bounded. This can be proved by induction.

For, if (α, β) denote any black interval of any of the countable aggregate of sub-sets of Harnack points used in defining the integral, and α_n and β_n *any* two points in the completely open interval (α, β), which as $n \to \infty$ approach α and β respectively, we have by definition

$$\int_\alpha^\beta f(x)\,dx = \operatorname*{Lt}_{n\to\infty} \int_{\alpha_n}^{\beta_n} f(x)\,dx,$$

the asterisk having been suppressed.

Suppose that $F(x)$ is known to be continuous at every point of the completely open interval (α, β). Then we have also

$$\int_{\alpha'}^\beta f(x)\,dx = \operatorname*{Lt}_{n\to\infty} \int_{\alpha'_n}^{\beta_n} f(x)\,dx.$$

Hence, subtracting,

$$F(\alpha')-F(\alpha) = \operatorname*{Lt}_{n\to\infty} \int_{\alpha'_n}^{\beta_n} f(x)\,dx - \operatorname*{Lt}_{n\to\infty} \int_{\alpha_n}^{\beta_n} f(x)\,dx.$$

Now we can choose α' so near α, and α'_n so near α', that α'_n is as near as we please to α, and therefore the first integral on the right differs from $\int_\alpha^\beta f(x)\,dx$ by as little as we please. The second integral on the right has $\int_\alpha^\beta f(x)\,dx$ as limit. Hence, α' being sufficiently near to α, $F(\alpha')-F(\alpha)$ is numerically as small as we please. Thus $F(x)$ is continuous at the point α, and similarly it is continuous at β.

But $F(x)$ is known to be continuous at every point of a completely open black interval of the Harnack set; thus it is continuous throughout each such closed black interval. Hence it is continuous at every point of a completely open black interval of the first derived set H', and therefore throughout each closed black interval of H', and so on. In each black interval of each of the derived and deduced sets of the Harnack set, $F(x)$ is therefore continuous. In particular this is true in each closed black interval of the Harnack nucleus.

30. Let us then assume that G is the final set of the aggregate of defining sets, and is a perfect set, and that $F(x)$ is known to be continuous in its closed black intervals. We have seen that this is true when G is the Harnack nucleus.

Then, if x is a point of G, we have as before (§ 25)

$$F(x+h)-F(x)=\left[\int_G f(x)\,dx+\Sigma\int_{a_n}^{b_n} f(x)\,dx\right]_x^{x+h},$$

showing, as we already remarked, that, if $x+h$ is a point of the perfect set which moves up to x as limit, $F(x+h)-F(x)$ has the unique limit zero. If, however, $x+h$ is a point not belonging to G, we have as in § 25 an extra term to consider, which is not one of the terms of the unconditionally convergent series of integrals over the black intervals of G. This extra term is of the form $\int_{x+k}^{x+h} f(x)\,dx$, where $x+k$ is that one of the end-points of the black interval in which $x+h$ lies, which falls between x and $x+h$. Since, as $h\to 0$, the index n of this black interval must increase indefinitely, in order that the interval itself may become as small as we please, our extra term, being numerically $\leqslant W_n$, has zero as limit. Thus when this condition is satisfied, $F(x)$ is continuous at every point of G, and therefore continuous throughout the interval considered.

Thus $F(x)$ is continuous, provided the Harnack nucleus is the final set; or, when this is not the case, $F(x)$ is continuous in every black

interval of H_{N-S}, provided the condition $W_n \to 0$ holds for the Harnack nucleus.

Hence, by continued induction, the continuity of $F(x)$ is in all cases apparent.

Thus, summing up, we may say that *the necessary and sufficient conditions that a bounded Y-integral $F(x)$ should be a continuous function of x, are*

(i) *That in the formulæ defining the integral the asterisks may all be suppressed*, so that, when $\int_\alpha^\beta f(x)$ is defined as the unique limit of $\int_{\alpha_n'}^{\beta_n'} f(x)\,dx$, where α_n' and β_n' are points of (α_n, β_n) which, as $n \to \infty$, move up to α and β, the points α_n' and β_n' are not constrained by any other condition; and

(ii) *If G is the final set of the defining aggregate in the whole interval (a, b), or in any one of the black intervals (a_n, b_n) of any one of the sets of the defining aggregate, and W_n denotes the upper bound of the modulus of the Y-integral of $f(x)$ in any interval inside the n-th black interval of G (supposed arranged in any manner), then $W_n \to 0$, when $n \to \infty$.*

It may be noticed that *these conditions, like the conditions for the existence of the Y-integral, are countable in number*, the black intervals of all the sets of the defining aggregate forming a countable set. At the same time our investigation shows that *when $F(x)$ is continuous the condition* (ii) *is equally true when G is any perfect set, with respect to which $f(x)$ is summable, and such that the increments of $F(x)$ over its black intervals form an unconditionally convergent series.*

31. The preceding investigation shows that we may divide up Y-integrals into four different classes, each of which is supposed to contain the succeeding classes :—

(*a*) *those which have infinite discontinuities;*

(*b*) *those which are bounded without being necessarily continuous;*

(*c*) *those which are continuous;*

(*d*) *those which possess the Harnack-Moore-Denjoy property, and are accordingly also continuous.*

The class (*c*) is characterised by the conditions given in the preceding article, and the class (*d*) by the same conditions, together with the additional conditions that ΣW_n always converges; these additional conditions

of course, render the explicit enunciation of the condition (ii) of § 30 superfluous.

As Moore has virtually shown, the condition that ΣW_n should converge is precisely equivalent to the requirement that $\sum_{n=1}^{\infty} \int_{a_n+e}^{b_n-e'} f(x)\,dx$ should have a unique triple limit, as $n \to \infty$, $e \to 0$, $e' \to 0$.

32. Examples have already been given of Y-integrals of the classes (b) and (c). The following is an example of one belonging to class (a).

Ex. 3.—Let $$f(x) = n/(b_n - a_n) \qquad [a_n \leqslant x < \tfrac{1}{2}(a_n + b_n)],$$

and $$f(x) = -n(b_n - a_n) \qquad [\tfrac{1}{2}(a_n + b_n) \leqslant x \leqslant b_n],$$

where (a_1, b_1), (a_2, b_2), ... are the black intervals of a perfect set G of content zero.

Here $F(x)$ is zero at all the points of the perfect set G, and in each closed black interval of G is continuous, indeed between each end-point of a black interval and the middle point it is linear, and has the value $\frac{1}{2}n(a_n + b_n)/(b_n - a_n)$ at the middle point of (a_n, b_n).

Thus at each point of G it has an infinite discontinuity, on both sides, or on one side, according as the point is not, or is, an end-point of a black interval. At each point x of G which is not an end-point of a black interval, we have

$$\bar{F}(x+0) = +\infty, \quad \underline{F}(x+0) = 0, \quad \bar{F}(x-0) = 0, \quad \underline{F}(x-0) = -\infty,$$

and the same relations hold for an end-point of a black interval on the side opposed to the black interval.

33. We now turn to the sum theorem, which asserts that, if $f_1(x)$ and $f_2(x)$ are two functions, the integral of their sum is the sum of their integrals, and ask when this theorem holds, the integrals of $f_1(x)$ and $f_2(x)$ being Y-integrals?

Let $$h(x) = f_1(x) + f_2(x).$$

Then, if both functions are summable, $h(x)$ is also summable, and the sum theorem holds.

Hence we easily see that it always is true if one of the functions, say $f_2(x)$, is summable. For, if d' is any interval in which the sum theorem holds, that is

$$\int_{d'} h(x)\,dx = \int_{d'} f_1(x)\,dx + \int_{d'} f_2(x)\,dx, \tag{1}$$

and we let the interval d' expand in any way to d as limit, the second integral on the right, being continuous, will have $\int_d f_2(x)\,dx$ as unique limit. Hence, if the first integral on the right also has a unique limit, so will $\int_{d'} h(x)\,dx$, and the latter limit will be the sum of $\int_d f_1(x)\,dx$ and $\int_d f_2(x)\,dx$, if the expansion is such that the first integral on the right has $\int_d f_1(x)\,dx$ as limit.

Hence the sum theorem holds when f has only a finite number of Harnack points, and, by induction, when any of the derived or deduced sets of the Harnack set G_1 of f_1 consists of a finite number of points. Thus, *when $f_1(x)$ has only a countable set of Harnack points, and $f_2(x)$ is summable, the sum theorem holds.*

By the same argument it holds in any black interval d_n of the Harnack nucleus of $f_1(x)$, so that

$$\int_{d_n} h(x)\,dx = \int_{d_n} f_1(x)\,dx + \int_{d_n} f_2(x)\,dx. \tag{2}$$

But if d_n denote any interval in which (2) holds, and we let d_n describe any non-overlapping set of intervals, the second integral on the right will generate an unconditionally convergent series, therefore if the first integral on the right does so also, so will the integral on the left. If then the intervals d_n are the black intervals of any perfect set E over which $f_1(x)$ is summable, $h(x)$ is also summable over E, and

$$\int_E h(x)\,dx = \int_E f_1(x)\,dx + \int_E f_2(x)\,dx, \tag{3}$$

whence

$$\int_E h(x)\,dx + \sum_{n=1}^{\infty} \int_{d_n} h(x)\,dx$$

$$= \int_E f_1(x)\,dx + \sum_{n=1}^{\infty} \int_{d_n} f_1(x)\,dx + \int_E f_2(x)\,dx + \sum_{n=1}^{\infty} \int_{d_n} f_2(x)\,dx.$$

This shows that, if $f_1(x)$ is summable with respect to its Harnack nucleus, and its integrals over the black intervals of that set form an unconditionally convergent series, the same is true for $h(x)$, and, by definition, we get,

by the equation just written down,

$$\int_a^b h(x)\,dx = \int_a^b f_1(x)\,dx + \int_a^b f_2(x)\,dx,$$

so that the sum theorem is true.

Hence, by induction, *the sum-theorem holds in all cases, provided* $f_2(x)$ *is a summable function.*

34. *Similarly the sum theorem holds when one of the functions, say* $f_2(x)$, *has only a finite number of Harnack points.* For, in this case, $\int f_2(x)\,dx$ is still continuous, so that the argument of the preceding article still holds when the Harnack points of $f_1(x)$ are countable.

On the other hand, when the Harnack set G_1 of $f_1(x)$ is more than countable, the equation (2) still holds for any black interval of the Harnack nucleus of $f_1(x)$, which is also the Harnack nucleus of $h(x)$, since the Harnack points of $f_2(x)$, being only finite in number, disappear from the Harnack set of $h(x)$ at the first derivation. Hence the subsequent analysis is the same as in the preceding article.

35. When however both functions have an infinite number of Harnack points, it is clear that the sum-theorem does not hold without some restriction.

Suppose, for instance, $f_1(x)$ had the points $\frac{1}{2}, \frac{1}{4}, \ldots, 2^{-n}, \ldots$ for Harnack points, and $f_2(x)$ had $\frac{1}{3}, \frac{1}{9}, \ldots, 3^{-n}, \ldots$ for Harnack points. Then, unless $\int_x^1 f_1(x)\,dx$ had a unique limit when the lower limit of integration passes along the combined sequence formed by both sets of Harnack points, and the same is true for $\int_x^1 f_2(x)\,dx$, it is evident that the sum theorem could not hold, since our definition of the Y-integral of $h(x)$, which, by (2), is equal to the sum of the Y-integrals of f_1 and f_2 in every interval $(x, 1)$, when x passes along the combined sequence, demands the uniqueness of both these limits.

If therefore $f_1(x)$ *is such that the sum-theorem holds for it in combination with* EVERY *other function having a Y-integral, it is clearly necessary that the definition of its Y-integral should involve no starred limits.**

* This is Denjoy's condition (2).

This condition is, however, not sufficient. Even the more stringent demand that the Y-integrals should be continuous is not sufficient for this purpose, as will be shown by an example (§ 37).

36. If both the functions $f_1(x)$ and $f_2(x)$ have an infinite set of Harnack points, we have seen that we must necessarily require that the asterisks in the formulæ defining their integrals should be taken to apply to the set G formed by their Harnack sets G_1 and G_2.

Conversely, if this condition holds, $h(x)$ will satisfy the condition (i) of § 24 for any interval (a, b) in which $h(x)$ has an integral given by the sum-theorem. The condition (ii) is in any case satisfied. For if E is any perfect set, $f_1(x)$ is not totally non-summable with respect to it ; therefore there is an interval d_1, containing points of E, such that $f_1(x)$ is summable over E in d_1. Since $f_2(x)$ is not totally non-summable with respect to this sub-set of E, there is an interval d_2 inside d_1 containing points of E, such that $f_2(x)$, as well as $f_1(x)$, is summable over E in d_2. Therefore also $h(x)$ is summable over E in d_2. Thus $h(x)$ is not totally non-summable with respect to E, so that $h(x)$ satisfies the condition (ii) of § 24.

Thus it only remains to investigate the condition (iii) of § 24. This condition, however, only enters if one of the two functions has a more than countable set of Harnack points. Thus we see at once that *the sum theorem holds for any two functions with countable sets of Harnack points, provided the condition regarding the asterisks is satisfied.*

But we can make the same assertion whatever one of the functions may be, provided the other has only a countable Harnack set. For in any black interval of the Harnack nucleus of the former function, say $f_1(x)$, the sum theorem holds, by what has just been pointed out, since both functions have only countable sets of Harnack points in such an interval. Let then E denote this Harnack nucleus, or any other perfect set in whose black intervals the sum-theorem holds. The points of E which are Harnack points for the second function form a countable closed set, and if x be any other point of E, we can find an interval with x as internal point, containing none of these Harnack points. In this interval $f_2(x)$ is accordingly summable. Hence the sum of the integrals of $f_2(x)$ over the black intervals of E in the interval is unconditionally convergent. Since then over the black intervals of the sets to be considered the integrals of $f_1(x)$ form an unconditionally convergent series, the same is true of the sum of the integrals of $f_1(x)$ and $f_2(x)$, and therefore also of the integrals of $h(x)$. Thus all the points of E are not J points for $h(x)$. This proves, by induction, that the condition (ii) is satisfied, so that $h(x)$ possesses a Y-integral, and that this integral is given by the sum-theorem.

In particular if $f_1(x)$ and $f_2(x)$ both have continuous Y-integrals, and one of them possesses only a countable set of Harnack points, their sum $f_1(x)+f_2(x)$ has also a continuous Y-integral, and it is equal to the sum of their Y-integrals.

37. The following example shows that *mere continuity of the Y-integrals is not sufficient to ensure the truth of the sum-theorem.* At the same time it gives us a function $h(x)$ which does not possess a Y-integral.

Ex. 4.—Let $f(x)$ be the function defined in Ex. 2 (§ 27), having a continuous Y-integral. Denoting the middle point of the interval (a_n, b_n) by c_n, so that

$$c_n = \tfrac{1}{2}(b_n+a_n),$$

let $g(x)$ be defined in each interval (c_n, b_n) by the same rule as $f(x)$ was defined in (0, 1), and be zero elsewhere. The Harnack points of $g(x)$ consist then of Cantor's typical ternary set, and a set similar to that set in each of the right-hand halves of its black intervals. The content of this whole set is therefore again zero, and in each of the black intervals of this set the integral of $g(x)$ is zero; therefore

$$\int_a^b g(x)\,dx = \int_a^b f(x)\,dx = 0.$$

Here again $g(x)$ has a continuous Y-integral. But, writing

$$h(x) = f(x)+g(x),$$

the Harnack points of $h(x)$ are the same as those of $g(x)$. The black intervals of this Harnack set consist partly of the intervals (a_n, c_n), such that

$$\int_{a_n}^{c_n} h(x)\,dx = \int_{a_n}^{c_n} 1/m(b_n-a_n)\,dx = 1/2m,$$

and partly of intervals which we may denote by $(\alpha_{n,s}, \beta_{n,s})$, such that

$$\begin{aligned}\int_{\alpha_{n,s}}^{\beta_{n,s}} h(x)\,dx = &\int_{\alpha_{n,s}}^{\frac{1}{2}(\beta_{n,s}+\alpha_{n,s})} [-1/m(b_n-a_n)+1/r(\beta_{n,s}-\alpha_{n,s})]\,dx\\ &+\int_{\frac{1}{2}(\alpha_{n,s}+\beta_{n,s})}^{\beta_{n,s}} [-1/m(b_n-a_n)-1/r(\beta_{n,s}-\alpha_{n,s})]\,dx\\ &= -(\beta_{n,s}-\alpha_{n,s})/m(b_n-a_n)\\ &= -1/2m.3^r,\end{aligned}$$

where r depends on s by the same rule as m on n.

Thus the integrals of $h(x)$ over the black intervals of its Harnack points form a conditionally convergent series, whose terms are of the two forms $1/2m$ and $-1/2m.3^r$, there being, for each integer m, 2^{m-1} terms equal to $1/2m$, and, for each ensemble of the two integers (m, r), 2^{m+r-2} terms equal to $-1/2m.3^r$.

Moreover, as in § 27, there is no interval containing as internal point one of the Harnack points of $h(x)$, in which these integrals form an unconditionally convergent series. Thus $h(x)$ has no Y-integral.

38. Next suppose that both the functions $f_1(x)$ and $f_2(x)$ have a more than countable set of Harnack points, G_1 and G_2 respectively, G being the set consisting of G_1 and G_2 together, and let us suppose the necessary condition respecting the asterisks to be satisfied. Then, as we have seen, we only need to consider the convergence of the series of integrals which occur. If then the nuclei of G_1 and G_2 are identical, the sum-theorem holds. For if E be any of the perfect sets which occur, it is a sub-set of the common Harnack nucleus, so that we can assign an interval d, containing points of the perfect set E, in which both

$$\Sigma \int_{a_n}^{b_n} f_1(x)\,dx \quad \text{and} \quad \Sigma \int_{a_n}^{b_n} f_2(x)\,dx$$

converge unconditionally. Hence, supposing the sum-theorem to hold in the black intervals (a_n, b_n) of the perfect set E, $\Sigma \int_{a_n}^{b_n} h(x)\,dx$ also converges unconditionally in the interval d. This proves that $h(x)$ has not a J-point at every point of E, so that the condition (ii) of § 24 is satisfied. Thus $h(x)$ possesses a Y-integral, and by induction the sum-theorem holds, since, by our preceding results, it holds in the black intervals of the common Harnack nucleus.

Thus *the sum-theorem holds whenever the two functions have the same Harnack nucleus, provided they satisfy the condition respecting the asterisks; in particular this will be the case if their integrals are continuous.*

39. In general when the Harnack sets G_1 and G_2 of the two functions $f_1(x)$ and $f_2(x)$ are both more than countable, the nuclei of these two sets will not coincide, and will form together the nucleus of the set G, consisting itself of G_1 and G_2 together. Assuming that the condition of § 35

respecting the asterisks is satisfied, we shall, as we saw, only have to examine the various perfect sets which occur in the analysis of the set G with reference to the function

$$h(x) = f_1(x) + f_2(x).$$

The sum-theorem holding, as we have seen, in the black intervals of the nucleus of G, the discussion will proceed stage by stage, exactly parallel to that given in discussing the existence of the Y-integral, the conditions (i) and (ii) of § 24, being, as we saw satisfied. It follows that the only condition required to ensure the truth of the sum-theorem is that corresponding to (iii). *This condition is that there should be no perfect sub-set of G consisting entirely of J points for $f_1(x)$ or $f_2(x)$.*

To state this more at length, let E be any perfect sub-set of G, and E_1 the common part of E and G_1. Then we know that the integrals of $f_1(x)$ over the black intervals of any perfect sub-set of E_1 converge unconditionally in some interval containing a point of E_1; the same is true of E_1 itself, if it is perfect. But the existence of the integral of f_1 involves no condition of this kind with regard to the integrals of f_1 over the black intervals of E. Our condition is equivalent to assuming that there is some interval containing any point of E, in which these integrals converge unconditionally.

In particular this will be the case if the Harnack-Moore condition is satisfied, which demands that ΣW_n should converge, where W_n is the upper bound of the modulus of the integral of $f_1(x)$ over any interval inside the n-th black interval of the set E, when these are arranged in any chosen order, E being any perfect set whatever. Less restrictedly it will be also the case if the same is true when E is any perfect sub-set of G. Denjoy's condition that, whatever perfect set E may be, there is some interval containing points of E in which ΣW_n converges, is similarly sufficient for the truth of the sum-theorem.

Conversely, it is evident that, *if we refuse to relax the demand that the sum of two functions having integrals is to have an integral, given by the sum of their integrals, we are bound to restrict the term integral to Denjoy integrals.* For, if $f(x)$ is a function not satisfying Denjoy's condition, there must be some perfect set E, such that in every interval containing a point of the set the series W_n does not converge. Let (a_1, b_1), (a_2, b_2), ... be the black intervals of E, then, by the meaning of W_n, there is inside each interval (a_n, b_n) an interval (a_n', b_n'), such that

$$W_n - 2^{-n} \leqslant \int_{a_n'}^{b_n'} f(x)\,dx \leqslant W_n + 2^{-n}.$$

The series of integrals $\left|\int_{a_n'}^{b_n'} f(x)dx\right|$ is then also divergent. Hence, if we attempt to apply the sum-theorem to $f(x)$, and a function $g(x)$ having the points a_n', b_n', for all integers n, among the points of its Harnack nucleus, we are stopped by the fact that the condition of the present article is not fulfilled.

40. Having now defined the most general Y-integrals and discussed some of their properties, we pass to the application of them to the theory of trigonometrical series. We have first the following theorem :—

THEOREM.—*If $f(x)$ possesses a bounded Y-integral $F(x)$, then the integrated series of the Fourier series of $f(x)$, where the term Fourier series is used in the usual sense, except that Y-integrals take the place of ordinary integrals in the expressions for the coefficients of the series, is the Fourier series of the bounded function $F(x)$, and accordingly converges to $F(x)$, when summed in the Cesàro manner, at all points not belonging to at most a certain set of content zero.*

In fact, if $$\pi a_r = \int_0^{2\pi} f(x)\cos rx\,dx,$$

then applying the formula for integration by parts,

$$\pi a_r/r = -\int_0^{2\pi} F(x)\,d(\cos rx)/r + \Big[F(x)(\cos rx)/r\Big]_0^{2\pi}$$

$$= [F(2\pi)-F(0)]/r + \int_0^{2\pi} F(x)\sin rx\,dx = \int_0^{2\pi}[F(x)-\tfrac{1}{2}a_0x]\sin rx\,dx.$$

Similarly, $$\pi b_r/r = \int_0^{2\pi} f(x)\sin rx\,dx/r = -\int_0^{2\pi} F(x)\cos rx\,dx$$

$$= \int_0^{2\pi}[F(x)-\tfrac{1}{2}a_0x]\cos rx\,dx.$$

Thus $$F(x)-\tfrac{1}{2}a_0x \sim C + \sum_{r=1}^{\infty}(a_r\sin rx - b_r\cos rx)/r,$$

where C is a constant, depending on the particular indefinite integral $F(x)$ chosen.

COR.—*If a trigonometrical series is the Fourier series of a function $f(x)$ possessing a bounded Y-integral, it is the derived series of the Fourier*

series of a bounded function $\phi(x)$. This function $\phi(x)$ is, to a linear additive term près, the Y-integral of $f(x)$.

41. We shall require the following theorem :—

THEOREM.—*If $f(x)$ has a continuous Y-integral, and $g_n(x)$ is a bounded monotone decreasing (increasing) function of x which converges as $n \to \infty$ to a bounded function $g(x)$, necessarily monotone, then*

$$\operatorname*{Lt}_{n\to\infty} \int_a^b f(x)\, g_n(x)\, dx = \int_a^b f(x)\, g(x)\, dx.$$

First let the interval (a, b) be such that in any interval (a', b') contained in it, whose end-points are different from a or b, the theorem is true. In the first instance we may suppose that, in each such interval (a', b'), $f(x)$ has a Lebesgue integral. Then, by the known properties of a Lebesgue integral, the theorem is true in (a', b').

Since $f(x)$ has a Y-integral,

$$\int_{a'}^{b'} f(x)\, dx \to \int_a^b f(x)\, dx,$$

when a' moves up to a and b' to b in an appropriate manner. Since $F(x)$ is continuous, we can find a' so near a, and b' so near b, that if X_1 is any point between a and a', and X_2 between b' and b,

$$-e \leqslant \int_a^{X_1} f(x)\, dx \leqslant e, \quad -e \leqslant \int_{X_2}^b f(x)\, dx \leqslant e,$$

and therefore, by the Second Theorem of the Mean, supposing the functions $g_n(x)$ monotone decreasing,

$$-e g_n(a) \leqslant \int_a^{X_1} f(x)\, g_n(x)\, dx \leqslant e g_n(a),$$

and

$$-e g_n(b') \leqslant \int_{b'}^{X_2} f(x)\, g_n(x)\, dx \leqslant e g_n(b').$$

Hence

$$\int_{X_1}^{b'} f(x)\, g_n(x)\, dx - e[g_n(a) + g_n(b')] \leqslant \int_a^{X_2} f(x)\, g_n(x)\, dx$$

$$\leqslant \int_{X_1}^{b'} f(x)\, g_n(x)\, dx + e[g_n(a) + g_n(b')].$$

Now let X_2 move up to b: then the middle member becomes in the limit $\int_a^b f(x)g_n(x)dx$, and the extreme members of this inequality are unaltered.

But in the interval (X_1, b') the theorem holds, and therefore, letting $n \to \infty$, the extremes approach as unique limits the two quantities

$$\int_{X_1}^{b'} f(x)\,g(x)\,dx \pm e[g(a)+g(b')]. \tag{i}$$

The central member has one or more limits, denoted by

$$\underset{n\to\infty}{\text{Llt}} \int_a^b f(x)\,g_n(x)\,dx, \tag{ii}$$

which are independent of the positions of X_1 and b'. If therefore we now let X_1 move up to a and b' to b, these latter (ii) remain unaltered, and the extremes (i) in our inequality become

$$\int_a^b f(x)\,g(x)\,dx \pm e[g(a)+g(b-0)],$$

and differ accordingly from another, and from $\int_a^b f(x)\,g(x)\,dx$, by a quantity which is as small as we please. It follows that the limits (ii) must all coincide, and we get finally

$$\underset{n\to\infty}{\text{Lt}} \int_a^b f(x)\,g_n(x)\,dx = \int_a^b f(x)\,g(x)\,dx,$$

which proves the theorem in this case, when $g(x)$ is monotone decreasing and bounded. Hence by induction the theorem is true for any interval in which the Harnack points form a countable set.

If therefore, as usual, (a_r, b_r) denote the typical black interval of the Harnack nucleus corresponding to the Y-integral $\int_a^b f(x)dx$ in the general case, we have

$$\sum_{r=1}^{m} \underset{n\to\infty}{\text{Lt}} \int_{ar}^{b_r} f(x)\,g_n(x)\,dx = \sum_{r=1}^{m} \int_{a_r}^{b_r} f(x)\,g(x)\,dx.$$

Let $\phi_m(x)$ denote the function which is equal to $f(x)$ in these m black intervals and is zero elsewhere: then the last equation becomes

$$\underset{n\to\infty}{\text{Lt}} \int_a^b \phi_m(x)\,g_n(x)\,dx = \int_a^b \phi_m(x)\,g(x)\,dx. \tag{iii}$$

Now, since by hypothesis $F(x)$ is continuous, we can, having pre-assigned a positive quantity e, determine d so that the oscillation of $F(x)$ in an interval of length less than d is less than e; we can then determine m so large, that all black intervals $\geqslant d$ are among the chosen m black intervals. Also since the Y-integral exists, we know that the increments of $F(x)$ over the black intervals form an absolutely convergent series whose sum is $F(b)$; thus we may also choose m so large that the sum of the increments of $F(x)$ over the non-chosen intervals is numerically less than e.

Let us now denote by $\psi_m(x)$ the function which is equal to zero in the chosen intervals and equal to $f(x)$ elsewhere, so that

$$f(x) = \phi_m(x) + \psi_m(x). \tag{iv}$$

Then, by the Second Theorem of the Mean, supposing again $g_n(x)$ to be monotone decreasing

$$\int_a^b \psi_m(x)\, g_n(x)\, dx = g_n(a) \int_a^X \psi_m(x)\, dx + g_n(b) \int_X^b \psi_m(x)\, dx,$$

where X is a point of the closed interval (a, b). But the integral of $\psi_m(x)$ between a and any point of the perfect set P, is, by the definition of the integrand, the sum of the increments of $F(x)$ over the non-chosen intervals in the interval of integration, and is therefore numerically less than e. If therefore X is a point of the set P, the first term on the right of the last equation is numerically less than $|g_n(a)|\, e$. But if X is not a point of the set P, it is internal to one of the black intervals; if this is one of the chosen intervals, $\psi_m(x)$ is zero in it, and we have the same result as if X were the right-hand end-point of this black interval; thus in this case we have the same result. If finally X is internal to one of the non-chosen intervals, the increment of $F(x)$ from X to the right-hand end-point of that black interval is, since that interval is less than d, not greater than e; and this is also the integral of $\psi_m(x)$ from X to the right-hand end-point of the black interval. Adding this to $\int_a^X \psi_m(x)\, dx$, we get, therefore, again a quantity less than e; therefore $\left| \int_a^X \psi_m(x)\, dx \right| < 2e$. Thus, in all cases, the first term on the right of the last equation is numerically less than $2e\,|g_n(a)|$. Similarly the second term is numerically less than $2e\,|g_n(b)|$. Thus, finally,

$$\left| \int_a^b \psi_m(x)\, g_n(x)\, dx \right| \leqslant 2e\,[\,|g_n(a)| + |g_n(b)|\,]. \tag{v}$$

Proceeding to the limit with n, we have therefore

$$\operatorname*{Lt}_{n\to\infty}\left|\int_a^b \psi_m(x)\, g_n(x)\, dx\right| \leqslant 2e[\,|g(a)|+|g(b)|\,]. \tag{vi}$$

But, in the argument by which we obtained the relation (v), $g_n(x)$ might be replaced by any other bounded monotone decreasing function, in particular by $g(x)$, provided $g(x)$ is bounded. Thus

$$\left|\int_a^b \psi_m(x)\, g(x)\, dx\right| \leqslant 2e\,[\,|g(a)|+|g(b)|\,]. \tag{vii}$$

Now, by (iv),

$$\begin{aligned}&\left|\operatorname*{Lt}_{n\to\infty}\int_a^b f(x)\, g_n(x)\, dx-\int_a^b f(x)\, g(x)\, dx\right| \\ &\quad=\left|\operatorname*{Lt}_{n\to\infty}\int_a^b \phi_m(x)\, g_n(x)\, dx-\int_a^b \phi_m(x)\, g(x)\, dx\right. \\ &\qquad\qquad\left.+\operatorname*{Lt}_{n\to\infty}\int_a^b \psi_m(x)\, g_n(x)\, dx-\int_a^b \psi_m(x)\, g(x)\, dx\right| \\ &\quad\leqslant 4e\,[\,|g(a)|+|g(b)|\,],\end{aligned}$$

by (iii), (vi), and (vii). Since this last quantity is as small as we please, the left-hand side of the inequality must be zero, that is

$$\operatorname*{Lt}_{n\to\infty}\int_a^b f(x)\, g_n(x)\, dx=\int_a^b f(x)\, g(x)\, dx,$$

which proves the theorem in the general case, provided $g_n(x)$ and $g(x)$ are bounded and monotone decreasing.

Changing the signs of $g_n(x)$ and $g(x)$, therefore, this also proves the theorem when these functions are monotone increasing. Thus the theorem is true.

COR.—*If $F(x)$ be a continuous Y-integral, and $g_n(x)$ is a bounded monotone decreasing (increasing) function of x which converges to a bounded monotone function $g(x)$ as $n\to\infty$, then*

$$\int_a^b F(x)\, dg_n(x)\to\int_a^b F(x)\, dg(x).$$

This follows at once from our theorem, using integration by parts.

42. The preceding theorem may be applied to prove the following:—

THEOREM.—*If $f(x)$ possesses a continuous Y-integral, and $g(x)$ is a function of bounded variation, then the integral of the product $f(x)\,g(x)$, taken between any two limits, is equal to the Cesàro sum of the series obtained by integrating term-by-term the product of $g(x)$ and the Fourier series of $f(x)$, provided only that, if the length of the integral of integration exceeds 2π, $f(x)$ is a periodic function with period 2π.*

In fact, we know that

$$\begin{aligned}\tfrac{1}{2}[g(x+0)+g(x-0)] &= \operatorname*{Lt}_{n\to\infty} \frac{1}{2n\pi}\int_{-e}^{e} g(u+x)\left(\frac{\sin\frac{1}{2}nu}{\sin\frac{1}{2}u}\right)^2 du \\ &= \operatorname*{Lt}_{n\to\infty} \frac{1}{2n\pi}\int_{0}^{e} [g(u+x)-g(x-u)]\left(\frac{\sin\frac{1}{2}nu}{\sin\frac{1}{2}u}\right)^2 du \\ &= \operatorname*{Lt}_{n\to\infty} G(x,\,n) \\ &= \operatorname*{Lt}_{n\to\infty} G_1(x,\,n) - \operatorname*{Lt}_{n\to\infty} G_2(x,\,n),\end{aligned}$$

where, as $n\to\infty$, the functions $G_1(x,\,n)$ and $G_2(x,\,n)$ trace out two sequences of monotone functions of x, and the left-hand side is equal to $g(x)$, except at a set of content zero.

Hence, by the theorem of § 41, we have

$$\int_{-\pi}^{\pi} f(x)\,g(x)\,dx = \operatorname*{Lt}_{n\to\infty}\int_{-\pi}^{\pi} f(x)\,G(x,\,n)\,dx.$$

The argument is precisely the same as in the proof of Theorem 4, § 9, of my paper "On the Integration of Fourier Series" (*Proc. London Math. Soc.*, Ser. 2, Vol. 9, p. 458, 1910).

COR.—*In the enunciation of the theorem we may replace the Fourier series of $f(x)$ by the Fourier series of $g(x)$, and multiply it by $f(x)$, provided we require $g(x)$ to be periodic instead of $f(x)$, in the case of an interval of integration greater than 2π.*

In fact, in proving the theorem we have really proved the corollary first.

43. The preceding theorem may be even more simply proved as follows:—

Since $F(x)$, the Y-integral of $f(x)$, is a continuous function, its Fourier

series converges to $F(x)$ boundedly, and even uniformly, when summed in the Cesàro manner, except possibly at the ends of the interval of integration. We may then integrate term-by-term with respect to $g(x)$. By means of the theorem of integration by parts, the required result will then follow.

Cor. 2.—*The theorem is still true if $F(x)$ has only a countable set of discontinuities, provided only that $F(x)$ is bounded.*

The Cesàro summations of the Fourier series of $F(x)$ will still converge uniformly to $F(x)$, except at a countable set.

ON THE CLASSIFICATION OF THE INTEGRALS OF LINEAR PARTIAL DIFFERENTIAL EQUATIONS OF THE FIRST ORDER

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Introduction.

This investigation is limited to the case in which the partial differential equation determines a relation between one dependent and two independent variables.

With the usual notation the equation is

$$Pp+Qq=R, \tag{I}$$

and the subsidiary system of simultaneous Lagrangian equations is

$$dx/P=dy/Q=dz/R. \tag{II}$$

It will save space to call each of the ratios in (II), or any equivalent ratio derived from them, a Lagrangian ratio.

If any two independent integrals of (II) are

$$u=\text{const.}, \quad v=\text{const.}, \tag{III}$$

then denoting partial differential coefficients by suffixes, the equations

$$Pu_x+Qu_y+Ru_z\equiv 0, \quad Pv_x+Qv_y+Rv_z\equiv 0, \tag{IV}$$

are *identically* satisfied for all values of x, y, z.

It will be convenient to call the integrals u, v *ordinary* integrals of (I) to distinguish them from the *general* integral

$$\phi(u, v)=0.$$

Besides these integrals it has long* been known that other integrals of (I) exist, viz.:—

If $$w(x, y, z) = 0, \tag{V}$$

be such an integral, then $Pw_x+Qw_y+Rw_z$

does not vanish *identically* for all values of x, y, z; but only when x, y, z are connected by the relation (V).

This may be expressed by writing

$$Pw_x+Qw_y+Rw_z = W, \tag{VI}$$

where W is some quantity which vanishes when w vanishes.

Any integral of this kind will *for the purposes of this paper* be called a special integral. It will be shown that such integrals fall into two classes, one of which is not included, whilst the other is included, in the general integral.

If it should happen that either of the *ordinary* integrals becomes infinite when w vanishes, then it will in this paper be invariably replaced by its reciprocal or some other function of itself, which is finite when w vanishes. If this be not done then special integrals which really belong to the second class may appear to belong to the first class [see Examples 5 (ii), 21, 27].

The Jacobian $J = \dfrac{\partial(u, v, w)}{\partial(x, y, z)}$ is always calculated in this paper on the understanding that neither u nor v becomes infinite when w vanishes.

If $w = 0$ be a special integral, the Jacobian J does not vanish *identically* for all values of x, y, z. Sometimes it vanishes when w vanishes; sometimes it vanishes when some relation between x, y, z, other than $w = 0$, holds good; sometimes it does not vanish at all.

From (IV) and (VI) it follows that

$$P\Big/\frac{\partial(u, v)}{\partial(y, z)} = Q\Big/\frac{\partial(u, v)}{\partial(z, x)} = R\Big/\frac{\partial(u, v)}{\partial(x, y)} = W/J. \tag{VII}$$

* In Prof. Forsyth's Presidential Address to the London Mathematical Society (*Proc. London Math. Soc.*, Ser. 2, Vol. 4, 1905–6, p. 438) there occurs the following:—"It appears to me that there is a very definite need for a re-examination and a revision of the accepted classification of integrals of equations even of the first order: in the usual establishment of the familiar results, too much attention is paid to unspecified form, and too little attention is paid to organic character, alike of the equations and of the integrals. Also, it appears to me possible that, at least for some classes of equations, these special integrals may emerge as degenerate forms of some semi-general kinds of integrals; but it is even more important that methods should be devised for the discovery of these elusive special integrals."

Following Goursat and Forsyth I will denote the common value of the ratios in (VII) by the letter M.

The system* on which the integrals of the equation are classified in this paper is based on the value of the Jacobian J.

(i) If J vanish *identically* for all values of x, y, z, then w is one of the *ordinary* integrals.

(ii) If J be either a constant, or some function of x, y, z which does not vanish when w vanishes, then w will be called a special integral of the first class.

(iii) If J vanish when w vanishes, it will be shown that an *ordinary* integral exists which vanishes when w vanishes.

In this case w is included in this ordinary integral, but will, *for the purposes of this paper*, be called a special integral of the second class.

The line of demarcation between the two classes is sometimes a very narrow one, as will be seen on examining Examples 3 and 5, in which a certain locus belongs to one class for a certain range of values of a parameter contained in the equation, and to the other class for a certain other range of values of the same parameter.

In the case of those equations which have a special integral of the first class and which have one or more coefficients containing an irrational form, then in all the cases which I have examined the special integral is the same as the singular integral of the rationalised equation.

This investigation covers the following ground :—

(I) The construction of the most general form of equation which possesses special integrals.

(II) The proof that in every case in which special integrals exist they can be obtained by applying a suitable method of integration to the Lagrangian system of subsidiary equations.

(III) The classification of all the integrals of the equation.

(IV) Remarks on the application to the theory of Cauchy's theorem regarding the existence of the solution of the subsidiary Lagrangian equations.

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* A similar system may be constructed for the classification of the integrals of equations of the form

$$f(x, y)\frac{dy}{dx} = \phi(x, y).$$

(V)–(XI) These sections contain examples illustrating the various points of the theory. In order to show that this theory covers all known cases of special integrals I have included amongst the examples all those which I have been able to find in the work of other mathematicians, and to these I have added some of those which I have constructed, sufficient in number to illustrate the issues which arise.

I.

1. *To construct a linear partial differential equation of the first order which shall have*

$$w(x, y, z) = 0, \tag{VIII}$$

as a special integral.

It has been seen that the value of z given by (VIII) satisfies

$$Pp+Qq = R, \tag{I}$$

and also

$$Pw_x+Qw_y+Rw_z = W, \tag{VI}$$

where W vanishes when w vanishes.

Taking R from (I) and putting it into (VI) it follows that

$$P(w_x+pw_z)+Q(w_y+qw_z) = W. \tag{IX}$$

The sole restriction on P and Q is that they have no common factor which could be removed from the equation (IX) and so leave the right hand side of (IX) in a form which would not vanish when w vanishes.

There is a form which differs slightly from (IX), which is sometimes more convenient.

Let X, Y, Z be any functions of x, y, z; then the form is

$$P(w_x+pw_z)+Q(w_y+qw_z) = W(Xp+Yq-Z). \tag{X}$$

As it can be written

$$\left(P-\frac{WX}{w_z}\right)(w_x+pw_z)+\left(Q-\frac{WY}{w_z}\right)(w_y+qw_z)+\frac{W}{w_z}(Xw_x+Yw_y+Zw_z) = 0,$$

it is not essentially distinct from the form (IX).

II.

2. *To prove that any special integral of the equation*

$$Pp+Qq = R, \tag{I}$$

is included amongst the integrals of the corresponding Lagrangian system.

If a special integral w exist, it satisfies an equation of the form

$$Pw_x+Qw_y+Rw_z = W, \tag{VI}$$

where W vanishes when w vanishes.

Now the subsidiary Lagrangian system of (I) is

$$dx/P = dy/Q = dz/R. \tag{II}$$

Each of these Lagrangian ratios is equal to

$$(w_x dx+w_y dy+w_z dz)/(Pw_x+Qw_y+Rw_z) = dw/W.$$

Hence we may replace the Lagrangian system by

$$dx/P = dy/Q = dw/W.$$

Taking x as the independent variable*, these are equivalent to

$$dy/dx = Q/P, \qquad dw/dx = W/P.$$

Now it is possible to take $w = 0$ as a solution of the last equation, since W vanishes when w vanishes.

Then removing z by eliminating it between $w = 0$ and $dy/dx = Q/P$, we are left with an equation of the first order in x and y. The integral contains one arbitrary constant. This integral with $w = 0$ furnishes an integral system of the Lagrangian equations.

By hypothesis $w = 0$ is a special integral of (I), *and it has been shown that it can be obtained by a suitable method of integrating the Lagrangian equations.*

As soon as it becomes apparent that one of the Lagrangian ratios takes the form dw/W, where W vanishes when w vanishes, this ratio will in this paper be described as becoming indeterminate when $w = 0$, and it will be said that $w = 0$ is a special integral of (I).

3. Let us now take the form (X) as it brings out one point more clearly than does (IX).

Treating (X) as a linear partial differential equation to determine z in terms of x and y, the subsidiary Lagrangian equations are

$$\begin{aligned} dx/(Pw_z-XW) &= dy/(Qw_z-YW) = dz/(-Pw_x-Qw_y-ZW) \\ &= -(w_x dx+w_y dy+w_z dz)/\{W(Xw_x+Yw_y+Zw_z)\} \\ &= -dw/\{W(Xw_x+Yw_y+Zw_z)\}. \end{aligned} \tag{XI}$$

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* See footnote † on p. 224.

Consider first the case in which $Xw_x+Yw_y+Zw_z$ vanishes identically for all values of x, y, z.*

Then $w = \text{const.}$ is an *ordinary* integral of (X), and therefore $w = 0$ *is a particular form of an ordinary integral, and is therefore not a special integral.*

Illustrations of this case are given in Examples 29 and 30.

But if $Xw_x+Yw_y+Zw_z$ is not *identically* zero, then we can still find a solution of (X) as in Art. 2.

Observe first we may suppose that w_z does not vanish when w vanishes. For, if w_z vanish for all the values of x, y, z which make w vanish, then by taking a point on $w = 0$ near x, y, z, say $x+\delta x$, $y+\delta y$, $z+\delta z$, we must have $w+w_x\delta x+w_y\delta y+w_z\delta z$ also vanishing with w; therefore $w_x\delta x+w_y\delta y$ must vanish with w. Taking $\delta y = 0$ we must have $w_x = 0$, similarly taking $\delta x = 0$ we must have $w_y = 0$. If further in this case w be a rational function of x, y, z it will be an exact square. On the other hand, in such a case as when $w^2+wx = (x+y+z)^2$, all the values of x, y, z which make w vanish also make w_x, w_y, w_z vanish. We shall throughout this paper exclude all such cases from consideration. If in the calculation of J it happens that w is such a function, J vanishes with w *independently* of the values of u and v.

Next take x as the independent variable;† then we get from (XI),

$$dy/dx = (Qw_z-YW)/(Pw_z-XW), \qquad \text{(XII)}$$

$$dw/dx = -W(Xw_x+Yw_y+Zw_z)/(Pw_z-XW). \qquad \text{(XIII)}$$

The denominator occurring on the right hand side of both (XII) and (XIII) does not vanish with w, for w_z and P do not vanish with w.

Consequently $w = 0$ is a solution of (XIII).

Eliminating z between $w = 0$ and (XII), the latter takes the form

$$dy/dx = [Q/P], \qquad \text{(XIV)}$$

where $[Q/P]$ is a function of x and y.

The integral of (XIV) is a relation between x, y and an arbitrary constant. This integral together with $w = 0$ form an integral system of (XI).

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* This is the point which would not have appeared if we had taken the form (IX), because W does not vanish identically.

† If P vanish with w, then Q cannot also vanish with w, without making R vanish with w. Now if P, Q, R all vanish with w, then a factor vanishing with w can *in general* be removed from the equation. Suppose any such factor, if it exist, to have been removed. If then P vanish with w, we may in general suppose that Q does not. In this case, instead of taking x for independent variable, we take y.

Thus the integral $w = 0$ *of* (X) *can be obtained from the subsidiary Lagrangian system* (XI) *by integrating it in a suitable manner.*

On looking at the form (XI) it is seen that the integral $w = 0$ makes one of the Lagrangian ratios contained in it indeterminate.

III.

Classification of the Integrals of the Equation.

4. The system on which the integrals of the equation are classified is based on the value of the Jacobian J, in the calculation of which it is always to be remembered that the ordinary integrals are to be taken in a form which makes them finite when the special integral vanishes.

(i) If J vanish *identically* for all values of x, y, z, then w is a function of u and v, and is one of the ordinary integrals, being some expression of the form $\phi(u, v)$.

(ii) If J is either a constant or some function of x, y, z which does not vanish with w, then w will *for the purposes of this paper* be called a special integral of the first class. In this case w is not included in the form $\phi(u, v)$, nor is it possible to derive the values of z which make $w = 0$ from any equation of the form $\phi(u, v) = 0$. Goursat calls all integrals *included in* the form $\phi(u, v) = 0$ non-singular integrals, from which it may be inferred that he would regard an integral such as is now under consideration as a singular integral. Chrystal takes the same view. Forsyth, however, places all integrals *not actually expressible* in the form $\phi(u, v) = 0$ in the class of special integrals without distinction. From Forsyth's point of view there is no question in this case of the existence of a singular integral ; for the equation being a linear one cannot satisfy the conditions laid down by him (*Theory of Differential Equations*, Vol. 5, p. 184) for the existence of a singular integral.

In most of the examples of this kind which I have found or been able to construct, one or more of the coefficients were irrational, though irrationality in the coefficients is not by itself sufficient to make the integral a special integral of the first class. In all the cases considered the equations when rationalised satisfy Forsyth's conditions for the possession of a singular integral.

The integrals of the subsidiary Lagrangian equations when rationalised represent a congruence of curves. The Lagrangian equations can frequently be integrated without rationalising them. In these cases the integrals of the unrationalised equations represent portions of the curves of the congruence which terminate abruptly at a focal point. As a point moves along a curve of the congruence towards a focal point, the tangent

at the point to the curve in general tends to a position lying in the tangent plane to the focal surface. The focal surface is usually* the envelope of the *complete* curves of the congruence.

The characteristic of these special integrals of the first class is, from the geometrical point of view, the existence of a focal surface, which is at the same time the envelope of the congruence of curves representing the integrals of the rationalised Lagrangian equations.

I have also constructed equations having special integrals of the first class in which the coefficients are transcendental (see Examples 10, 11, and 12). All of these present points of interest, and the second and third some points of difficulty.

(iii) If J vanish when w vanishes, then w will *for the purposes of this paper* be called a special integral of the second class. It will be shown that in this case some form of the equation $\phi(u, v) = 0$ exists such that $\phi(u, v)$ vanishes when w vanishes. Hence in this case all the values of z which make $w = 0$ can be obtained from an equation of the form $\phi(u, v) = 0$, and therefore the special integral w is *included in* the general integral, although w is not itself of the form $\phi(u, v)$. Although these integrals are *included in* the general integral there is an advantage in placing them in a separate class, because they arise in the same way as do the special integrals of the first class, when the subsidiary system of Lagrangian equations is being integrated.

It is sometimes possible to guess a special integral of the second class when one or both the ordinary integrals cannot be obtained in any manageable form. Thus in the case of Example 16 (due to Forsyth), until I had succeeded in performing one of the two integrations involved, I did not suspect the relationship of the special integral given by Forsyth to the ordinary integral; and I supposed that the equation given in this example belonged to, and was evidence of the existence of, a class of equations having special integrals distinct from the two kinds mentioned above.

In Ex. 15, J vanishes when w vanishes, but $w = 0$ has the envelope property, like the special integrals of the first class.

5. It will now be shown that *if J vanishes when w vanishes, a form of $\phi(u, v)$ exists which vanishes when w vanishes.*

The following are the logical alternatives in regard to any ordinary integral u.

(i) u vanishes when w vanishes.

(ii) u is infinite when w vanishes.

(iii) u is finite when w vanishes.

* For cases of exception see Examples 8 and 29.

In the first case the proposition to be proved is admitted.

In the second case, by taking the reciprocal of u instead of u, we see that an ordinary integral exists, which vanishes when w vanishes.

In the third case, let us suppose that both u and v are finite when w vanishes, and that (by eliminating z) u and v are expressed as functions of x, y, and w.

Let D denote partial differentiation when x, y, and w are treated as independent variables.

Then
$$J = \frac{\partial(u, v, w)}{\partial(x, y, z)} = w_z \frac{D(u, v)}{D(x, y)}.$$

Hence, if J vanishes when w vanishes, either w_z or $\frac{D(u, v)}{D(x, y)}$ vanishes when w vanishes.

Now the case where w_z vanishes may be excluded for the reasons given in § 3.

Hence we have to consider what conclusions can be drawn from the vanishing of
$$\frac{D(u, v)}{D(x, y)}$$
when w vanishes.

Since u, v are finite when w vanishes, we may* suppose that

$$u = u_0 + f(w)\,\phi(x, y, w),$$

$$v = v_0 + g(w)\,\psi(x, y, w),$$

where u_0, v_0 are functions of x, y only; $f(w)$, $g(w)$ vanish when w vanishes; and $\phi(x, y, w)$, $\psi(x, y, w)$ are finite when w vanishes.

Then
$$\frac{Du}{Dx} = \frac{Du_0}{Dx} + f(w)\frac{D\phi(x, y, w)}{Dx}.$$

Hence when w vanishes,
$$\frac{Du}{Dx} = \frac{Du_0}{Dx}.$$

Similarly, $\frac{Du}{Dy} = \frac{Du_0}{Dy}$, $\frac{Dv}{Dx} = \frac{Dv_0}{Dx}$, $\frac{Dv}{Dy} = \frac{Dv_0}{Dy}$, when w vanishes; and therefore when w vanishes,
$$\frac{D(u, v)}{D(x, y)} = \frac{D(u_0, v_0)}{D(x, y)},$$

* If u and v are not undetermined when $w = 0$.

and so $\frac{D(u_0, v_0)}{D(x, y)}$ must vanish when w vanishes. But $\frac{D(u_0, v_0)}{D(x, y)}$ does not contain w at all. Therefore

$$\frac{D(u_0, v_0)}{D(x, y)} \equiv 0,$$

for all values of x, y. Therefore there is some identical relation between u_0, v_0, say

$$\Theta(u_0, v_0) \equiv 0.$$

Let us then form the integral

$$\Theta(u, v) = 0.$$

Putting in the values

$$u = u_0 + f(w)\,\phi(x, y, w),$$

$$v = v_0 + g(w)\,\psi(x, y, w),$$

the term independent of w is

$$\Theta(u_0, v_0),$$

which vanishes identically, and therefore the left-hand side of the equation $\Theta(u, v) = 0$ vanishes with w, since $f(w)$ and $g(w)$ both vanish with w.

Hence a form of the general integral exists which vanishes when w vanishes.

5a. In the cases in which J does not vanish with w it is possible to draw some conclusions of a partial nature about the coefficients P, Q, R. If we suppose that they are finite, *and not all zero*, when w vanishes, then because

$$Pw_x + Qw_y + Rw_z = W,$$

where W vanishes when w vanishes, but J does not vanish with w, then equations (VII) show that some of the denominators

$$\frac{\partial(u, v)}{\partial(y, z)}, \quad \frac{\partial(u, v)}{\partial(z, x)}, \quad \text{and} \quad \frac{\partial(u, v)}{\partial(x, y)},$$

must become infinite when $w = 0$.

This requires in general that either $\frac{df(w)}{dw}$ or $\frac{dg(w)}{dw}$ becomes infinite when $w = 0$.

[The case of exception, in which $w = 0$ makes P, Q, R all vanish, is expressly excluded (see Ex. 11).]

(A) Let us suppose that $f(w)$ vanishes like w^m, so that $m > 0$. Then, if $\frac{df(w)}{dw}$ is to be infinite when w vanishes, we must have m a positive

proper fraction. In this case in order to remove the irrational factor in the denominators of

$$\frac{\partial(u, v)}{\partial(y, z)}, \quad \frac{\partial(u, v)}{\partial(z, x)}, \quad \text{and} \quad \frac{\partial(u, v)}{\partial(x, y)},$$

we must multiply through by w^{1-m}. The effect of this is to make the *lowest* exponent of the irrational terms in P, Q, R a positive proper fraction. But it is not true conversely that, if the *lowest* exponent of the irrational terms in P, Q, R is a positive proper fraction, then the equation will have a special integral of the first class (see Exs. 15,* 29).

(B) I give a case of an equation (Ex. 10) in which $f(w)$ has the form $1/\log w$.

The complete primitive has in this case a sharp edge, along which it ends abruptly, and so resembles very closely the integrals of the non-rationalised equation discussed in Ex. 2.

As a point on the primitive approaches the sharp edge, the tangent plane tends to coincide with the special integral $w = 0$.

(C) I give a case of an equation (Ex. 11) in which $f(w)$ has the form $e^{-1/w}$, and w tends to $+0$. It presents at least one perplexing feature, which I can only set out.

As w passes from $+0$ to -0, the surface is discontinuous. $w = 0$ is a special integral, such as is recognised by Goursat and Chrystal, which makes $P = Q = R = 0$. The plane $w = 0$ meets but does not cut† every complete primitive in two straight lines; it is not a tangent plane to each complete primitive along these straight lines. Every point on $w = 0$ is a point at which the two sharp edges, belonging to *some* complete primitive, meet.

(D) I give an example (No. 12) in which $f(w)$ and $g(w)$ are undetermined when $w = 0$.

6. *If an ordinary integral can be written in the form*

$$st = \text{const.},$$

where $s = \text{const.}$, $t = \text{const.}$ *are not ordinary integrals, then* $s = 0$ *and* $t = 0$ *both satisfy the definition of a special integral if they give finite values for* z *in terms of* x *and* y.

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* See last sentence of § 4.

† Except possibly at the point of intersection of the two sharp edges.

In this case

$$P\frac{\partial(st)}{\partial x}+Q\frac{\partial(st)}{\partial y}+R\frac{\partial(st)}{\partial z}\equiv 0,$$

for all values of x, y, z. Therefore

$$t(Ps_x+Qs_y+Rs_z)+s(Pt_x+Qt_y+Rt)\equiv 0.$$

Now let x, y, z be chosen so as to make $s=0$, and let us suppose the factors s, t so chosen that the values of x, y, z which make $s=0$ do not make $t=0$. Then unless these values make $(Pt_x+Qt_y+Rt_z)/t$ infinite, we shall have $Ps_x+Qs_y+Rs_z=0$ when $s=0$. Hence $s=0$ satisfies the definition of a special integral.

Further, on the hypothesis that

$$\frac{1}{t}(Pt_x+Qt_y+Rt_z)$$

is finite, it follows that $Pt_x+Qt_y+Rt_z=0$ when $t=0$, and therefore $t=0$ is also a special integral.

Since $s=0$, $t=0$ are obtainable as particular cases of the ordinary integral $st=$ const., viz., by taking the constant to vanish, they are included in the general integral and are special integrals of the second class.

The value of the Jacobian J in this case is, if we put $v=st$,

$$\frac{\partial(u,v,s)}{\partial(x,y,z)}=\frac{\partial(u,st,s)}{\partial(x,y,z)}=s\frac{\partial(u,t,s)}{\partial(x,y,z)};$$

so that J vanishes when s vanishes, but not identically.

Special integrals obtained by the methods of §§ 6 and 8 may be regarded as *isolated* portions of particular forms of an ordinary integral.

7. If the values of z, which make $s=0$, make

$$\frac{1}{t}(Pt_x+Qt_y+Rt_z)$$

infinite, then since

$$\frac{1}{s}(Ps_x+Qs_y+Rs_z)+\frac{1}{t}(Pt_x+Qt_y+Rt_z)=0,$$

those values of z also make

$$\frac{1}{s}(Ps_x+Qs_y+Rs_z)$$

infinite. The following proposition is a contribution to the discussion of these values, though it does not cover all cases.

If the ordinary integrals u and v can be expressed as ALGEBRAIC *functions of x, y, z, and if $w = 0$ be a special integral such that*

$$Pw_x + Qw_y + Rw_z$$

vanishes like w^n; then, if $0 < n < 1$, $w = 0$ is a special integral of the first class; but if $n \geqslant 1$, $w = 0$ is a special integral of the second class.

Take, as in § 5, the ordinary integrals in the forms

$$u = u_0 + f(w)\,\phi(x, y, w),$$

$$v = v_0 + g(w)\,\psi(x, y, w),$$

where u_0, v_0 are functions of x, y only, and suppose that $f(w)$ vanishes like w^m, and $g(w)$ vanishes like w^{m+r} (so that $m > 0$, $r \geqslant 0$), whilst $\phi(x, y, w)$, $\psi(x, y, w)$ are finite when $w = 0$.

Further, if we write

$$\kappa = \frac{\partial(u_0, v_0)}{\partial(x, y)} + f(w)\frac{D(\phi, v_0)}{D(x, y)} + g(w)\frac{D(u_0, \psi)}{D(x, y)} + f(w)\,g(w)\frac{D(\phi, \psi)}{D(x, y)},$$

$$\lambda = \begin{vmatrix} \dfrac{\partial u_0}{\partial y} + f(w)\dfrac{D\phi}{Dy} & \dfrac{\partial v_0}{\partial y} + g(w)\dfrac{D\psi}{Dy} \\ f'(w)\,\phi + f(w)\dfrac{D\phi}{Dw} & g'(w)\psi + g(w)\dfrac{D\psi}{Dw} \end{vmatrix},$$

$$\mu = \begin{vmatrix} \dfrac{\partial u_0}{\partial x} + f(w)\dfrac{D\phi}{Dx} & \dfrac{\partial v_0}{\partial x} + g(w)\dfrac{D\psi}{Dx} \\ f'(w)\,\phi + f(w)\dfrac{D\phi}{Dw} & g'(w)\psi + g(w)\dfrac{D\psi}{Dw} \end{vmatrix},$$

then

$$\frac{\partial(u, v)}{\partial(y, z)} = \lambda w_z, \qquad \frac{\partial(u, v)}{\partial(z, x)} = -\mu w_z,$$

$$\frac{\partial(u, v)}{\partial(x, y)} = \kappa - \lambda w_x + \mu w_y, \qquad J = \kappa w_z.$$

Referring to equations (VII) it is seen that P, Q, R, and W are obtained from $\dfrac{\partial(u, v)}{\partial(y, z)}$, $\dfrac{\partial(u, v)}{\partial(z, x)}$, $\dfrac{\partial(u, v)}{\partial(x, y)}$, and J respectively by multiplying them by a certain factor M.

The essential property of M is that it must contain as a factor such a

power of w as shall be just sufficient to prevent the occurrence of any negative power of w in P, Q, and R.

Now the exponent of the lowest power of w in λ and μ is $m-1$, and in κ it is zero.

It is necessary to consider separately the cases $0 < m < 1$ and $m \geqslant 1$.

(i) If $0 < m < 1$, it is essential that M should contain the factor w^{1-m}.

Hence we may take

$$Pw_x+Qw_y+Rw_z = Jw^{1-m} = \kappa w_z w^{1-m}.$$

Now the term containing the lowest power of w in $\kappa w_z w^{1-m}$ is the term

$$\frac{\partial(u_0, v_0)}{\partial(x, y)} w_z w^{1-m},$$

and the next power of w which occurs is given by the term

$$f(w) \frac{D(\phi, v_0)}{D(x, y)} w_z w^{1-m},$$

and since $f(w)$ vanishes like w^m, this term contains the first power of w.

So that, if $0 < m < 1$, the lowest power of w which can occur in $Pw_x+Qw_y+Rw_z$ has $1-m$ for its exponent, and the exponent of the next term in w is unity.

(ii) If however $m \geqslant 1$, then the lowest power of w which can occur in λ and μ is the $(m-1)$-th, and the lowest power in κ has zero index. As in this case there are no powers of w with negative indices in κ, λ, μ, we may take $M = 1$. Therefore

$$Pw_x+Qw_y+Rw_z = J = \kappa w_z.$$

Now in order that $w = 0$ may be a solution of the equation,

$$Pw_x+Qw_y+Rw_z$$

must vanish when w vanishes, and therefore J must vanish when w vanishes. But this requires that

$$\frac{\partial(u_0, v_0)}{\partial(x, y)} = 0,$$

and consequently the lowest power of w in $Pw_x+Qw_y+Rw_z$ is given by the term

$$f(w) \frac{D(\phi, v_0)}{D(x, y)} w_z,$$

and has the exponent m, which is $\geqslant 1$.

Putting the two cases together we see that, if $0 < m < 1$, the lowest power of w which can occur in $Pw_x+Qw_y+Rw_z$ has for its exponent a positive proper fraction, and the exponent of w next in order of magnitude is unity; whilst if $m \geqslant 1$, then the lowest power of w which can occur in $Pw_x+Qw_y+Rw_z$ has an exponent which is $\geqslant 1$.

Now suppose it is given that $Pw_x+Qw_y+Rw_z$ vanishes like w^n, and suppose first that $0 < n < 1$. This can only happen when $0 < m < 1$, and $n = 1-m$, and in order that this case may actually occur, $\frac{\partial(u_0, v_0)}{\partial(x, y)}$ must not vanish when w vanishes, and therefore J must not vanish when w vanishes, and therefore w is a special integral of the first class.

Suppose next that $n \geqslant 1$. This may happen either when $0 < m < 1$ or $m \geqslant 1$. If $0 < m < 1$, it can only happen when

$$\frac{\partial(u_0, v_0)}{\partial(x, y)} = 0;$$

and therefore J vanishes when w vanishes. Whilst if $m \geqslant 1$ it has been proved that J vanishes when w vanishes. Hence in both cases in which $n \geqslant 1$, J vanishes when w vanishes; and therefore $w = 0$ is a special integral of the second class.

Ex. 3 and 5 are illustrations of the proposition just proved.

The proposition has been expressly limited to the case in which the ordinary integrals are expressible as *algebraic* functions of x, y, z.

For if, as in Ex. 10, $f(w)$ and $g(w)$ vanish like $1/\log w$, then $f'(w)$ and $g'(w)$ contain the infinite term $1/\{w(\log w)^2\}$ when $w = 0$. To remove the infinite terms in λ and μ it is necessary that M should contain the factor w.*

If we take $M = w$, then

$$Pw_x+Qw_y+Rw_z = wJ = w\kappa w_s,$$

and so, if J do not vanish when w vanishes, then the lowest power of w which occurs in $Pw_x+Qw_y+Rw_z$ is the first,† and $w = 0$ is a special integral of the first class because J does not vanish when w vanishes.

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* In Ex. 10 the form of the equation can be more conveniently written if M be taken as $w(\log w)^2$, and it is for this reason that this value of M is taken in constructing the equation, but M might have been taken equal to w.

† When the ordinary integrals were algebraic it was shown that if the lowest power of w which occurred in $Pw_x+Qw_y+Rw_z$ were the first, then $w = 0$ was a special integral of the second class.

If, however, J vanishes when w vanishes, then $Pw_x+Qw_y+Rw_z$ vanishes like $wf(w)$, *i.e.*, like $w/\log w$, and $w=0$ is a special integral of the second class.

8. *If an ordinary integral can be written in the form*

$$s/t = \text{const.},$$

where $s = \text{const.}$, $t = \text{const.}$ *are not ordinary integrals, then* $s=0$ *and* $t=0$ *both satisfy the definition for a special integral if they give finite values for* z *in terms of* x *and* y.

For
$$P\frac{\partial(s/t)}{\partial x}+Q\frac{\partial(s/t)}{\partial y}+R\frac{\partial(s/t)}{\partial z}\equiv 0,$$

for all values of x, y, z. Therefore

$$t(Ps_x+Qs_y+Rs_z)-s(Pt_x+Qt_y+Rt_z)=0.$$

Arguing as in § 6, it follows that both $s=0$ and $t=0$ are special integrals of the second class.

The case here considered is really a particular case of § 6, for

$$s/t = \text{const.},$$

can be written
$$s\times\left(\frac{1}{t}\right) = \text{const.},$$

or
$$t\times\left(\frac{1}{s}\right) = \text{const.},$$

and then § 6 can be applied.

If we are considering the special integral $s=0$, then we take

$$J=\frac{\partial(u,\,s/t,\,s)}{\partial(x,\,y,\,z)}=\frac{s}{t^2}\,\frac{\partial(u,\,s,\,t)}{\partial(x,\,y,\,z)}.$$

But if we are considering $t=0$, then we take

$$J=\frac{\partial(u,\,t/s,\,t)}{\partial(x,\,y,\,z)}=-\frac{t}{s^2}\,\frac{\partial(u,\,s,\,t)}{\partial(x,\,y,\,z)}.$$

It is important to observe this distinction, for if we were considering $s=0$, and took for J the value

$$J=\frac{\partial(u,\,t/s,\,s)}{\partial(x,\,y,\,z)}=\frac{1}{s}\,\frac{\partial(u,\,t,\,s)}{\partial(x,\,y,\,z)},$$

J would not vanish with s in general, and the special integral would

appear to belong to the first class, whereas it really belongs to the second class.

The Envelope Property.

9. Taking the forms $u = u_0 + f(w)\,\phi(x, y, w)$,

$$v = v_0 + g(w)\,\psi(x, y, w),$$

for the ordinary integrals, $w = 0$ being the special integral, u_0 and v_0 functions of x, y only, $f(w)$ and $g(w)$ vanishing when $w = 0$, but $f'(w)$ and $g'(w)$ being infinite when $w = 0$, and D denoting partial differentiation with regard to x, y, w; then we have

$$u_x = \frac{\partial u_0}{\partial x} + f(w)\frac{D\phi}{Dx} + \left(f'(w)\,\phi(x, y, w) + f(w)\frac{D\phi}{Dw}\right) w_x,$$

$$u_y = \frac{\partial u_0}{\partial y} + f(w)\frac{D\phi}{Dy} + \left(f'(w)\,\phi(x, y, w) + f(w)\frac{D\phi}{Dw}\right) w_y,$$

$$u_z = \left(f'(w)\,\phi(x, y, w) + f(w)\frac{D\phi}{Dw}\right) w_z.$$

Then since $f'(w)$ is infinite when $w = 0$, we shall have, when $w = 0$, u_x, u_y, u_z proportional to the terms which contain $f'(w)$, and therefore proportional to w_x, w_y, w_z.

Hence at any point where the surface $w = 0$ meets any surface $u = \text{const.}$, these two surfaces touch one another.

Hence $w = 0$ is usually an envelope of the surfaces $u = \text{const.}$, and similarly it is an envelope of the surfaces $v = \text{const.}$ In some of the examples given the only points where $w = 0$ meets an ordinary integral are at infinity. In such cases $w = 0$ is not to be regarded as an envelope.

It should be observed that though the constituents u_x, u_y, u_z of the Jacobian J are proportional to the constituents w_x, w_y, w_z when $w = 0$, it does not follow that J vanishes when $w = 0$.

For on calculating J its value is found to be

$$w_z \left\{ \frac{\partial(u_0, v_0)}{\partial(x, y)} + f(w)\frac{D(\phi, v_0)}{D(x, y)} + g(w)\frac{D(u_0, \psi)}{D(x, y)} + f(w)\,g(w)\frac{D(\phi, \psi)}{D(x, y)} \right\},$$

which when $w = 0$ becomes $w_z \dfrac{\partial(u_0, v_0)}{\partial(x, y)}$, and therefore does not in general vanish.

In Ex. 11, $f(w) = e^{-1/w}$, $f'(w) = w^{-2}e^{-1/w}$,

so that when $w = +0$, $f'(w) = 0$, and so the above argument does not apply. In this case the special integral does not possess the envelope property, as is pointed out in the discussion of the example.

Other Properties of the Jacobian.

10. It follows from equations (VII) that

$$P\frac{\partial J}{\partial x} + Q\frac{\partial J}{\partial y} + R\frac{\partial J}{\partial z} = M\frac{\partial(u, v, J)}{\partial(x, y, z)}.$$

Consequently—

(*a*) $P\frac{\partial J}{\partial x} + Q\frac{\partial J}{\partial y} + R\frac{\partial J}{\partial z}$ will vanish identically if J is a constant or a function of u and v.

J is a constant in Exs. 2, 3 (if $0 < n < 1$), 4, and 9.

J is a function of u and v in Exs. 1, 3 (if $n \geqslant 1$), 18, 19, 21, 22, 23, 25, 26, 27, and 28.

These two cases are closely connected, for if J be a function of u and v, and we take a function $\chi(u, v)$, such that

$$\frac{\partial \chi(u, v)}{\partial u} = 1/J,$$

then the Jacobian of $\chi(u, v)$, v, w with regard to x, y, z is unity; and provided that $\chi(u, v)$ is not infinite when $w = 0$, we can take this new Jacobian in place of the original one.

(*b*) $P\frac{\partial J}{\partial x} + Q\frac{\partial J}{\partial y} + R\frac{\partial J}{\partial z}$ vanishes if J vanishes in two cases,

(i) If M vanishes when J vanishes.

This happens in Ex. 15, and J is a special integral of the second class.

(ii) If $\frac{\partial(u, v, J)}{\partial(x, y, z)}$ vanishes with J. Then J is a special integral of the second class.

This happens in Exs. 7, 10 (ii) and (iii), 11, 12, 13, 14, 16, 17, 20 and 24.

(*c*) $P\frac{\partial J}{\partial x} + Q\frac{\partial J}{\partial y} + R\frac{\partial J}{\partial z}$ may not vanish either identically, or conditionally with J.

This happens in the case of Exs. 5, 6, and 8.
In this case J is not an integral of the equation at all.
In Exs. 7, 10, 11, and 12, J is a function of w.

IV.

Remarks on the Application of Cauchy's Theorem, regarding the Existence of a Solution of the Subsidiary System of Lagrangian Equations, to the Theory.

11. Both Goursat and Forsyth give demonstrations of the comprehensiveness of the general integral based on the use of Cauchy's theorem above referred to.*

Goursat excludes from his proof any equation in which the special integral under consideration makes $P = Q = R = 0$. I give two illustrations of this kind of case. In one of these, obtained by transforming Chrystal's equation (Ex. 2), a form arises from which it is possible to remove a factor containing no differential coefficients, the effect of which procedure is to deprive the equation of the possession of a special integral.

In the second illustration (Ex. 11), the coefficients are transcendental, and all vanish when $z-x-y$ tends to $+0$. The special integral in this case is $z-x-y=0$. Some difficulties arise in connection with this example, which are set forth in the discussion.

12. Both Goursat and Forsyth point out that the validity of their proofs does not extend to cases where the coefficients of the equation are not regular at points on the special integral. But they do not enter upon any discussion as to the existence or non-existence of converse theorems, *i.e.*, they do not show that if the restrictions they lay down are not satisfied, then the special integral cannot be included in the general integral.

I am not in a position to formulate any general propositions regarding the dependence of the nature of the special integrals on the nature of the coefficients, but the following points are of interest in this connection.

(A) Nine examples (13, 14, 16–22) are given in which the coefficients are algebraic, one-valued and finite for all finite values of the variables. These equations have special integrals of the second class. If the special integral be $w = 0$, then in these examples W vanishes like some positive

* Goursat, *Leçons sur l'intégration des équations aux dérivées partielles du premier ordre*, p. 34 (1891); Forsyth, *Treatise on Differential Equations*, 4th ed., p. 400.

integral power of w. The Lagrangian equations belong to that kind in which the only value which Cauchy's theorem furnishes for the dependent variable w within a certain domain of $w = 0$ is the constant value zero. I have called special attention to this point in the three Exs. 18, 21, and 22.

(B) Two examples (Nos. 29 and 30) are given, showing that if the coefficients are irrational or non-regular in some other way, it does not necessarily follow that the equations have a special integral of either class. The locus of the critical or non-regular points in these cases is a particular form (not an isolated part of a particular form) of one of the ordinary integrals. In these cases J vanishes identically. M does not vanish on the special integral.

V.

FIRST GROUP OF EXAMPLES (NOS. 1–9).

Special Integrals of the First Class.

In this set of examples a focal surface exists.

The equations in all the cases considered after being rationalised satisfy the conditions for the existence of a singular solution.

J does not vanish on the special integral, but M does.

EXAMPLE 1.

The first example I take is from Goursat's *Leçons sur l'intégration des équations aux dérivées partielles du premier ordre*, p. 35 (1891). I think it is a legitimate inference from his work to say that he regarded it as having a singular integral. Goursat does not discuss the example. He merely gives it as having an integral which is not included in the general integral, pointing out that the points on this integral are critical points of one of the coefficients of the partial differential equation, so that it is a case in which one of the restrictions which he had laid down in the immediately preceding work is not observed.

It is interesting historically in that I believe it was the first example of the kind that was specially noticed.

The equation is

$$p(x^2+z^2-1)+q\{xy+(1-z^2)^{\frac{1}{2}}(x^2+y^2+z^2-1)^{\frac{1}{2}}\} = 0.$$

The ordinary integrals are

$$u \equiv \{xy+(1-z^2)^{\frac{1}{2}}(x^2+y^2+z^2-1)^{\frac{1}{2}}\}/(x^2+z^2-1) = \text{const.},$$

$$v \equiv z = \text{const.}$$

Goursat points out that $x^2+y^2+z^2-1=0$ is an integral not included in the general integral $\phi(u, v) = 0$.

I proceed to show how it can be derived from the subsidiary Lagrangian equations without first rationalising them.

Taking x as the independent variable, these equations are

$$dy/dx = \{xy+(1-z^2)^{\frac{1}{2}}(x^2+y^2+z^2-1)^{\frac{1}{2}}\}/(x^2+z^2-1),$$

$$dz/dx = 0.$$

From which it follows that

$$d(x^2+y^2+z^2-1)/dx$$

$$= 2(x^2+y^2+z^2-1)^{\frac{1}{2}}\{x(x^2+y^2+z^2-1)^{\frac{1}{2}}+y(1-z^2)^{\frac{1}{2}}\}/(x^2+z^2-1).$$

None way of solving these equations is to take $x^2+y^2+z^2-1=0$ as a solution of the last.

Substituting in the first it follows that

$$dy/dx = -x/y,$$

therefore $$x^2+y^2 = \text{const.}$$

But $$x^2+y^2+z^2-1 = 0;$$

therefore $$z = \text{const.},$$

which satisfies $$dz/dx = 0.$$

Thus all the Lagrangian equations are satisfied by

$$x^2+y^2+z^2-1=0 \quad \text{and} \quad z = \text{const.}$$

So that the integral $x^2+y^2+z^2-1=0$ of the linear partial differential equation is included amongst the integrals of the subsidiary Lagrangian system.

I will now show how it may be obtained from the complete primitive of the Lagrangian system.

One integral is $z = \text{const.} = a$, say. Therefore

$$dy/dx = \{xy+(1-a^2)^{\frac{1}{2}}(x^2+y^2+a^2-1)^{\frac{1}{2}}\}/(x^2+a^2-1).$$

From which it follows that

$$d\{xy+(1-a^2)^{\frac{1}{2}}(x^2+y^2+a^2-1)^{\frac{1}{2}}\}/dx$$
$$= 2x\{xy+(1-a^2)^{\frac{1}{2}}(x^2+y^2+a^2-1)^{\frac{1}{2}}\}/(x^2+a^2-1);$$

therefore $xy+(1-a^2)^{\frac{1}{2}}(x^2+y^2+a^2-1)^{\frac{1}{2}} = b(x^2+a^2-1).$

On rationalising and removing the factor (x^2+a^2-1), which does not contain b, there remains

$$(y-bx)^2 = (1-a^2)(1+b^2),$$

which, with $z = a$, constitutes the complete primitive of the Lagrangian system.

Replacing b by $\tan a$, the complete primitive is equivalent to

$$y\cos a - x\sin a = \pm(1-a^2)^{\frac{1}{2}}, \quad z = a;$$

and so consists of the congruence of lines which are the tangents to the circles represented by

$$x^2+y^2 = 1-a^2, \quad z = a.$$

Now in order to find the focal points of the congruence determined by the equations

$$f(x, y, z, a, b) = 0, \qquad \phi(x, y, z, a, b) = 0,$$

it is necessary to combine these equations with

$$\frac{D(f, \phi)}{D(a, b)} = 0.$$

Hence in this case the focal points are given by

$$(y-bx)^2-(1-a^2)(1+b^2) = 0, \quad z = a, \quad x(y-bx)+b(1-a^2) = 0;$$

and are therefore

$$x = \pm b(1-a^2)^{\frac{1}{2}}(1+b^2)^{-\frac{1}{2}}, \quad y = \mp(1-a^2)^{\frac{1}{2}}(1+b^2)^{-\frac{1}{2}}, \quad z = a.$$

Hence the focal surface is $x^2+y^2+z^2 = 1.$

The same result would be obtained if we were to start from the equations of the congruence, and vary the parameters a, b in such a manner that the forms of dy/dx and dz/dx are unaltered.

The Jacobian of u, v, $x^2+y^2+z^2-1$, with regard to x, y, z, is

$$2[x(x^2+y^2+z^2-1)^{\frac{1}{2}}+y(1-z^2)^{\frac{1}{2}}]^2/(x^2+z^2-1)^2. = 2(u^2+1),$$

and M is

$$(x^2+y^2+z^2-1)^{\frac{1}{2}}(x^2+z^2-1)^2/\{x(x^2+y^2+z^2-1)^{\frac{1}{2}}+y(1-z^2)^{\frac{1}{2}}\},$$

so that M vanishes, but J does not vanish on the special integral.

Example 2.

The next example is due to Chrystal, and was published in a paper entitled "A Demonstration of Lagrange's Rule for the Solution of a Linear Partial Differential Equation, with some Historical Remarks on Defective Demonstrations hitherto current," in the *Transactions of the Royal Society of Edinburgh*, Vol. xxxvi (1892). Chrystal says, *l.c.*, p. 556, referring to his demonstration: "It therefore affords a rigorous demonstration of Lagrange's rule so far as that rule is correct. It would be vain to extend the demonstration to singular solutions, because solutions of this kind can be produced for which Lagrange's rule is not true."

He then gives the example I am about to discuss, and says that the solution he gives "cannot be put into the Lagrangian form at all. . . . The solution is therefore a *singular solution* of the linear equation. It sometimes happens that the solutions classified as singular for the purposes of the above demonstration are particular or limiting cases of a Lagrangian equation; but this is not the case in the example just given."

I do not think Chrystal would have written the above if he had been aware that the solution of the equation which he gives could be obtained by applying a suitable method of integration to the subsidiary Lagrangian equations.

Chrystal's equation is

$$\{1+(z-x-y)^{\frac{1}{2}}\}\,p+q=2.$$

Taking y as the independent variable the Lagrangian system is

$$dx/dy=1+(z-x-y)^{\frac{1}{2}},\quad dz/dy=2.$$

These do not satisfy the conditions laid down by Forsyth (*Theory of Differential Equations*, Vol. 3, p. 153) for the possession of a singular integral; but if the first equation be rationalised then Forsyth's conditions are satisfied.

It follows from the unrationalised equations that

$$d(z-x-y)/dy=-(z-x-y)^{\frac{1}{2}}.$$

Now, if $z-x-y=0$, we cannot divide out by $(z-x-y)^{\frac{1}{2}}$; and therefore

the solution, which first presents itself, is

$$z-x-y = 0.$$

Eliminating z by means of this, we get

$$dx/dy = 1\ ;$$

therefore $$x = y-c,$$

and then $$z = 2y-c.$$

We should have reached the same result if the sign of the radical in the differential equation had been changed.

Taking now the alternative $z-x-y \neq 0$, the integral of

$$d(z-x-y)/dy = -(z-x-y)^{\frac{1}{2}}$$

is $$2(z-x-y)^{\frac{1}{2}}+y = \text{const.}$$

This has to be combined with

$$z-2y = \text{const.}$$

Hence the general integral is

$$\phi\{2(z-x-y)^{\frac{1}{2}}+y,\ z-2y\} = 0.$$

A complete primitive is

$$2(z-x-y)^{\frac{1}{2}}+y = \alpha(z-2y)+\beta.$$

This equation cannot be rationalised without obliterating the distinction between the equations

$$\{1+(z-x-y)^{\frac{1}{2}}\}\,p+q = 2, \quad \text{and} \quad \{1-(z-x-y)^{\frac{1}{2}}\}\,p+q = 2.$$

If a rigid distinction is to be drawn between these two equations, then the integral of one is

$$2(z-x-y)^{\frac{1}{2}} = \alpha z-(1+2\alpha)\,y+\beta,$$

and of the other $$-2(z-x-y)^{\frac{1}{2}} = \alpha z-(1+2\alpha)\,y+\beta.$$

The two equations represent the two halves of the parabolic cylinder

$$4(z-x-y) = \{\alpha z-(1+2\alpha)\,y+\beta\}^2,$$

the line of division between the halves being the line of contact with the tangent plane

$$z-x-y = 0,$$

one half lying on the positive side of the plane

$$az-(1+2a)\,y+\beta=0,$$

and the other on its negative side.

We reach the same result if we start from the equations

$$2\,(z-x-y)^{\frac{1}{2}}+y=a, \quad z-2y=b,$$

and after rationalising the first equation apply the process for finding the focal points of the congruence.

These are given by the equations

$$4\,(z-x-y)-(a-y)^2=0, \quad z-2y-b=0, \quad a-y=0.$$

Hence they are $x=a+b, \quad y=a, \quad z=2a+b$;

and the focal surface is $z-x-y=0.$

The direction of the tangent at the focal point

$$x=a+b, \quad y=a, \quad z=2a+b$$

to the half parabola represented by

$$2\,(z-x-y)^{\frac{1}{2}}+y=a, \quad \text{and} \quad z-2y=b,$$

is given by the equations

$$dx/\{1+(z-x-y)^{\frac{1}{2}}\}=dy/1=dz/2,$$

when we put $x=a+b, \quad y=a, \quad z=2a+b$;

and so the tangent is determined by the equations

$$dx/1=dy/1=dz/2,$$

and therefore lies in the plane $z-x-y=0$. Consequently if a point on the half parabola move towards a focal point, the tangent at the point moves to coincidence with a line in the plane $z-x-y=0$. This plane is the envelope of the complete parabolas.

The Jacobian of $2\,(z-x-y)^{\frac{1}{2}}+y$, $z-2y$, $z-x-y$ with regard to x, y, z, is -1, and

$$M=(z-x-y)^{\frac{1}{2}}.$$

It is perhaps worth while observing in connection with the example, that if we put

$$(z-x-y)^{\frac{1}{2}}=\xi,$$

it becomes

$$\xi\left[2(1+\xi)\frac{\partial\xi}{\partial x}+2\frac{\partial\xi}{\partial y}+1\right]=0,$$

and if we then remove the factor ξ, on the ground that it contains no differential coefficients, the character of the equation is entirely altered.

The solution of the Lagrangian system is now

$$x = a - 2\zeta - \zeta^2, \qquad y = b - 2\zeta,$$

and there is no singular solution left.

The factor ξ rejected, if equated to zero, gives such a singular solution as is recognised by Goursat and Chrystal.

EXAMPLE 3.

Consider the equation

$$\{1+(z-x-y)^n\}\, p+q = 2,$$

where n is a positive rational number. The Lagrangian equations are

$$dx/\{1+(z-x-y)^n\} = dy/1 = dz/2 = -d(z-x-y)/(z-x-y)^n.$$

Consequently $z-x-y=0$ is a special integral.

The ordinary integrals are

$$z-2y = a, \qquad y+\{(z-x-y)^{1-n}/(1-n)\} = b,$$

or if $n=1$, $\qquad y+\log(z-x-y) = b.$

If $0<n<1$, and $1-n = r/s$, so that $r<s$, then $z-x-y=0$ is a focal surface, and the special integral belongs to the first class.

If $n \geqslant 1$, there is no focal surface.

If $n=1$, and we take the second ordinary integral in the form

$$e^y(z-x-y) = \text{const.},$$

then, by § 6, $z-x-y=0$ is a special integral of the second class.

If $n>1$, and $n-1 = r/s$, then we can take the second integral in the form

$$(z-x-y)^{r/s}/\{ry(z-x-y)^{r/s}-s\} = \text{const.}$$

Then, by § 8, it follows that

$$z-x-y=0 \quad \text{and} \quad ry(z-x-y)^{r/s}-s=0$$

are special integrals of the second class.

If $0<n<1$, then, with the above forms of the integrals,

$$J=1 \quad \text{and} \quad M=-(z-x-y)^n.$$

If $n=1$, and $v = e^y(z-x-y)$, then $J=v$, and so vanishes on the special integral. In this case $M=-e^{-y}$.

If $n > 1$, and $v = (n-1)(z-x-y)^{n-1}/\{(n-1)\,y(z-x-y)^{n-1}-1\}$,

then $J = -v^2$, and so vanishes on the special integral. In this case

$$M = \{(n-1)\,y\,(z-x-y)^{n-1}-1\}^2/\{(n-1)^2\,(z-x-y)^{n-2}\}.$$

Example 4.

$$(1+\xi^{\frac{1}{2}}-\xi^{\frac{3}{2}})\,p+q = 2, \quad \text{where} \quad \xi = z-x-y.$$

The Lagrangian equations are

$$dx/(1+\xi^{\frac{1}{2}}-\xi^{\frac{3}{2}}) = dy/1 = dz/2 = -d\xi/(\xi^{\frac{1}{2}}-\xi^{\frac{3}{2}}).$$

This shows that $\xi = 0$ is a special integral.

The ordinary integrals are

$$z-2y = a, \qquad y-\log\frac{1-\xi^{\frac{1}{2}}}{1+\xi^{\frac{1}{2}}} = b,$$

and $J = 1$, showing that the special integral is of the first class, whilst

$$M = -\xi^{\frac{1}{2}}+\xi^{\frac{3}{2}}.$$

If we examine the nature of the second *ordinary* integral near $\xi = 0$, we have

$$y+2\xi^{\frac{1}{2}}\left(1+\frac{\xi}{3}+\frac{\xi^2}{5}+\ldots\right) = b;$$

therefore

$$4\xi\left(1+\frac{\xi}{3}+\frac{\xi^2}{5}+\ldots\right)^2 = (b-y)^2,$$

showing that $\xi = 0$ represents a focal surface.

Example 5.

Consider the equation

$$a\,(w_x+pw_z)+b\,(w_y+qw_z) = w^n,$$

where n is positive and rational, a and b are constants, and w is any given function of x, y, and z.

Since n is positive, $w = 0$ is a solution of the equation.

The Lagrangian equations are

$$dx/(aw_z) = dy/(bw_z) = dz/(-aw_x-bw_y+w^n)$$

$$= (w_x dx+w_y dy+w_z dz)/(w^n w_z) = dw/(w^n w_z).$$

Hence the Lagrangian equations are equivalent to

$$dx/a = dy/b = dw/w^n.$$

(i) Suppose $0 < n < 1$. Then the ordinary integrals are

$$bx - ay = \text{const.} = c, \quad (x/a) - \{w^{1-n}/(1-n)\} = \text{const.} = c'.$$

Now $1-n$ is a positive proper rational fraction. Let $1-n = r/s$, so that $r < s$.

Then $\frac{x}{a} - \frac{s}{r} w^{r/s} = c'$, therefore $w^r = \left[\frac{r}{s}\left(\frac{x}{a} - c'\right)\right]^s$.

Hence the focal points are given by

$$bx - ay = c, \quad w^r = \left[\frac{r}{s}\left(\frac{x}{a} - c'\right)\right]^s, \quad 0 = -r\left[\frac{r}{s}\left(\frac{x}{a} - c'\right)\right]^{s-1}.$$

Since the smallest value of s is 2, the focal points are given by

$$x = ac', \quad y = bc' - (c/a), \quad w = 0.$$

Hence the focal surface is $w = 0$, and

$$J = w_z, \quad M = w^n.$$

Here w is a special integral of the first class.

(ii) Suppose $n = 1$, $u = bx - ay$.

One form of v is $\frac{x}{a} - \log w$,

which becomes infinite when $w = 0$.

Hence we must take for v either

$$we^{-x/a}, \quad \text{or} \quad 1\Big/\left(\frac{x}{a} - \log w\right).$$

The form of J corresponding to

$$v = \frac{x}{a} - \log w,$$

is w_z, which does not vanish with w; so that it would appear that the special integral is of the first class. But the form of v taken is inadmissible in the calculation of J.

If we take $v = we^{-x/a}$,

then $J = -ww_z e^{-x/a}$,

and vanishes with w. If we take

$$v = 1 \Big/ \left(\frac{x}{a} - \log w\right),$$

then

$$J = w \Big/ \left(\frac{x}{a} - \log w\right)^2$$

which vanishes with w.

In this case there is no focal surface, and the special integral is of the second class.

(iii) If $n > 1$, then $v = \dfrac{x}{a} + [1/\{(n-1)\, w^{n-1}\}]$,

but this makes v infinite when $w = 0$. Hence, in accordance with the understanding stated on p. 220, we take for v the reciprocal of this value, and write

$$v = a(n-1)\, w^{n-1} / \{(n-1) x w^{n-1} + a\}.$$

And now $J = -v^2 w_z$, which vanishes when $w = 0$. Hence $w = 0$ is a special integral of the second class.

EXAMPLE 6.

$$\{3 + 2(z+x+2y)^{\frac{1}{2}}\}\, p + \{2(z-x-y)^{\frac{1}{2}} - 2\}\, q$$
$$= 1 + 2(z+x+2y)^{\frac{1}{2}} - 4(z-x-y)^{\frac{1}{2}} - 4(z-x-y)^{\frac{1}{2}}(z+x+2y)^{\frac{1}{2}}.$$

The ordinary integrals are

$$u \equiv (z-x-y)^{\frac{1}{2}} + x = \text{const.}, \qquad v = (z+x+2y)^{\frac{1}{2}} + y = \text{const.}$$

The equation can be thrown into the standard form of § 2 in the following ways:—

(i) Taking $w = (z-x-y)(z+x+2y)$, it is

$$\{3 + 2(z+x+2y)^{\frac{1}{2}}\}(w_x + p w_z) + \{2(z-x-y)^{\frac{1}{2}} - 2\}(w_y + q w_z)$$
$$= w^{\frac{1}{2}} \{4(z-x-y)^{\frac{1}{2}} - 6(z+x+2y)^{\frac{1}{2}} - 8z - 4y\}.$$

(ii) Taking $w = z+x+2y$, it is

$$\{3 + 2(z+x+2y)^{\frac{1}{2}}\}(w_x + p w_z) + \{2(z-x-y)^{\frac{1}{2}} - 2\}(w_y + q w_z)$$
$$= 4w^{\frac{1}{2}} \{1 - (z-x-y)^{\frac{1}{2}}\}.$$

(iii) Taking $w = z-x-y$, it is

$$\{3 + 2(z+x+2y)^{\frac{1}{2}}\}(w_x + p w_z) + \{2(z-x-y)^{\frac{1}{2}} - 2\}(w_y + q w_z)$$
$$= -2w^{\frac{1}{2}} \{3 + 2(z+x+2y)^{\frac{1}{2}}\}.$$

We can therefore take the two planes

$$z+x+2y=0, \quad z-x-y=0,$$

separately as special integrals. They are the two sheets of the associated focal surface.

I will first obtain them directly from the Lagrangian equations without making use of the complete primitive of those equations.

The Lagrangian system is

$$dx/\{3+2(z+x+2y)^{\frac{1}{2}}\} = dy/\{-2+2(z-x-y)^{\frac{1}{2}}\}$$
$$= dz/\{1+2(z+x+2y)^{\frac{1}{2}}-4(z-x-y)^{\frac{1}{2}}-4(z-x-y)^{\frac{1}{2}}(z+x+2y)^{\frac{1}{2}}\}.$$

Each of these ratios is equal to

$$-d(z-x-y)/[2(z-x-y)^{\frac{1}{2}}\{2(z+x+2y)^{\frac{1}{2}}+3\}]$$
$$= d(z+x+2y)/[4(z+x+2y)^{\frac{1}{2}}\{1-(z-x-y)^{\frac{1}{2}}\}].$$

Hence we may take either $z-x-y=0$ or $z+x+2y=0$ as a solution.

Take first $$z+x+2y=0.$$

This reduces the Lagrangian system to

$$dx/3 = dy/\{-2+2(-2x-3y)^{\frac{1}{2}}\} = d(-2x-3y)/\{-6(-2x-3y)^{\frac{1}{2}}\}. \quad .$$

Now we can integrate

$$dx/3 = d(-2x-3y)/\{-6(-2x-3y)^{\frac{1}{2}}\},$$

in two ways.

We can take $$x+(-2x-3y)^{\frac{1}{2}} = \text{const.},$$

which with $z+x+2y=0$ gives a solution of the Lagrangian system containing one arbitrary constant.

Or we may take the solution $2x+3y=0$, which, combined with $z+x+2y=0$, gives

$$y=-2x/3 \quad \text{and} \quad z=x/3.$$

This is a solution of the Lagrangian system containing no arbitrary constant.

The straight line represented by the equations

$$y=-\tfrac{2}{3}x, \quad z=\tfrac{1}{3}x,$$

is the intersection of the plane

$$z+x+2y=0,$$

with the plane $$z-x-y=0,$$

which represents the other special integral.

In like manner, if we had started with the special integral

$$z-x-y=0,$$

we should have been led to the same result.

To find the congruence of lines represented by the ordinary integrals when rationalised, we have the equations

$$z-x-y=(a-x)^2, \quad z+x+2y=(b-y)^2.$$

To obtain the focal points, we must combine these with the equation

$$(a-x)(b-y)=0.$$

If we take $x=a$, we have

$$z=y+a, \quad 2a+3y=(b-y)^2.$$

Hence there will be two focal points on the plane $z-x-y=0$. And therefore $z-x-y=0$ is one part of the focal surface.

Similarly, if we take $y=b$, we get

$$z+x+2b=0, \quad 2x+3b+(a-x)^2=0.$$

Hence there will be two focal points on the plane $z+x+2y=0$. And therefore $z+x+2y=0$ is one part of the focal surface.

The Jacobian of u, v and $z-x-y$ is $\frac{3}{2}(z+x+2y)^{-\frac{1}{2}}+1$.

The Jacobian of u, v and $z+x+2y$ is $-(z-x-y)^{-\frac{1}{2}}+1$.

The Jacobian of u, v and $(z-x-y)(z+x+2y)$ is

$$(z-x-y)-(z-x-y)^{\frac{1}{2}}+(z+x+2y)+\tfrac{3}{2}(z+x+2y)^{\frac{1}{2}}.$$

The value of M is $-4(z-x-y)^{\frac{1}{2}}(z+x+2y)^{\frac{1}{2}}$.

Example 7.

$$\{x(z-x-y)^{\frac{1}{2}}-y\}\,p+\{y(z-x-y)^{\frac{1}{2}}-x\}\,q$$
$$=2(z-x-y)^{\frac{1}{2}}\{1-(z-x-y)\}+(x+y)\{(z-x-y)^{\frac{1}{2}}-1\}.$$

The ordinary integrals are

$$u\equiv x+y(z-x-y)^{\frac{1}{2}}, \quad v\equiv y+x(z-x-y)^{\frac{1}{2}}.$$

The Lagrangian equations are

$$dx/\{x(z-x-y)^{\frac{1}{2}}-y\}$$
$$= dy/\{y(z-x-y)^{\frac{1}{2}}-x\}$$
$$= dz/[2(z-x-y)^{\frac{1}{2}}\{1-(z-x-y)\}+(x+y)\{(z-x-y)^{\frac{1}{2}}-1\}]$$
$$= d(z-x-y)/[2(z-x-y)^{\frac{1}{2}}\{1-(z-x-y)\}].$$

Thus $z-x-y=0$ and $z-x-y=1$

are special integrals

$$J = 1-(z-x-y) \quad \text{and} \quad M = 2(z-x-y)^{\frac{1}{2}}.$$

(i) Hence, for the special integral

$$z-x-y=0,$$

$J=1$. Hence this integral is of the first class.

(ii) But, for the special integral

$$z-x-y=1,$$

$J=0$. Hence it is of the second class.

If we form the integral $u^2-v^2=0$,

it is $$(x^2-y^2)\{1-(z-x-y)\}=0.$$

Hence, by § 6, the integral, $z-x-y=1$, is of the second class.

EXAMPLE 8.

$$(1+x^2)p+(1+y^2)q=(1-z^2)^{\frac{1}{2}}.$$

The Lagrangian equations are

$$dx/(1+x^2)=dy/(1+y^2)=dz/(1-z^2)^{\frac{1}{2}}.$$

Each of these is equal to $d(z\pm1)/(1-z^2)^{\frac{1}{2}}$, showing that $z+1=0$ and $z-1=0$ are special integrals. The ordinary integrals are

$$u=(x-y)/(1+xy)=a, \qquad v=\{y(1-z^2)^{\frac{1}{2}}-z\}/\{yz+(1-z^2)^{\frac{1}{2}}\}=b.$$

To obtain the focal points we have the equations

$$x-y=a(1+xy),$$

$$(1-z^2)(y-b)^2=z^2(1+by)^2, \qquad (1-z^2)(y-b)+yz^2(1+by)=0.$$

One way of satisfying these is to take

$$1-z^2=0, \quad 1+by=0, \quad x-y=a(1+xy).$$

Therefore the focal points are

$$x=\frac{ab-1}{a+b}, \quad y=-\frac{1}{b}, \quad z=\pm 1.$$

Hence the planes $1-z^2=0$ form a part of the focal surface, and their equation satisfies the differential equation.

Another way of satisfying the equations for the focal points is to take

$$z=0, \quad y=b, \quad x=\frac{a+b}{1-ab}.$$

Thus the plane $z=0$ is also a part of the focal surface. But $z=0$ does not satisfy the differential equation, whilst $1-z^2=0$ does satisfy it.

In this case $J=-2z(1+y^2)(1+xy)^{-2}(yz+\sqrt{1-z^2})^{-2}$,

$$M=+\sqrt{1-z^2}(1+y^2)^{-1}(1+xy)^2(yz+\sqrt{1-z^2})^2.$$

The special integral $1-z^2=0$ is of the first class.

Example 9.

$$\{1+(1+\xi^2)(\arctan\xi)^{\frac{1}{2}}\}\,p+q=2, \quad \text{where} \quad \xi=z-x-y.$$

The Lagrangian equations are

$$dx/\{1+(1+\xi^2)(\arctan\xi)^{\frac{1}{2}}\}=dy/1=dz/2=-d\xi/\{(1+\xi^2)(\arctan\xi)^{\frac{1}{2}}\},$$

showing that $\xi=0$ or $m\pi$, where m is an integer, is a special integral.

The ordinary integrals are

$$z-2y=a, \quad y+2\{\arctan(z-x-y)\}^{\frac{1}{2}}=b.$$

The focal points are given by the equations

$$z-2y=a, \quad 4\{\arctan(z-x-y)\}=(y-b)^2, \quad y-b=0,$$

i.e., by $\quad y=b, \quad z=a+2b, \quad z-x-y=m\pi,$

i.e., $\quad x=a+b-m\pi, \quad y=b, \quad z=a+2b.$

Here $\quad J=+1, \quad M=-(1+\xi^2)(\arctan\xi)^{\frac{1}{2}}.$

Here then the parallel planes

$$z-x-y = m\pi,$$

where m is an integer, form a special integral of the first class.

VI.

Second Group of Examples (*Nos.* 10–12).

Special integrals of the first class, in which the coefficients of the equation are transcendental, not irrational.

EXAMPLE 10.

$$[y-\{x/\log(z-x-y)\}](p-1)+[x-\{y/\log(z-x-y)\}](q-1)$$
$$= (z-x-y)[\{\log(z-x-y)\}^2-1].$$

The ordinary integrals are

$$u \equiv x+y/\{\log(z-x-y)\} = \text{const.}, \quad v \equiv y+x/\{\log(z-x-y)\} = \text{const.}$$

The Lagrangian equations are

$$dx/[y-\{x/\log(z-x-y)\}]$$
$$= dy/[x-\{y/\log(z-x-y)\}]$$
$$= dz/\Big[(x+y)\{1-[1/\log(z-x-y)]\}+(z-x-y)\{[\log(z-x-y)]^2-1\}\Big]$$
$$= d(z-x-y)/\Big[(z-x-y)\{[\log(z-x-y)]^2-1\}\Big]$$
$$= d(z-x-y-e)/\Big[(z-x-y)\{[\log(z-x-y)]^2-1\}\Big]$$
$$= d(z-x-y-e^{-1})/\Big[(z-x-y)\{[\log(z-x-y)]^2-1\}\Big].$$

Hence (i) $z-x-y=0$, (ii) $z-x-y=e$, (iii) $z-x-y=e^{-1}$ are special integrals.

$$J = 1-\{\log(z-x-y)\}^{-2}, \quad M = (z-x-y)\{\log(z-x-y)\}^2.$$

(i) The first special integral makes $J=1$, $M=0$, and is a special integral of the first class.

(ii) and (iii) The other two make $J=0$, M finite, and are special integrals of the second class. They are obtainable as factors of $u+v$ and $u-v$ by the method of § 6.

A complete primitive is

$$x+y/\{\log(z-x-y)\} = a\,[y+x/\{\log(z-x-y)\}]+b,$$

i.e.,
$$(x-ay-b)\log(z-x-y) = ax-y.$$

To study this put

$$x-ay-b = \xi, \quad ax-y = \eta, \quad z-x-y = \zeta.$$

Then the equation is
$$\xi = e^{\eta/\xi}.$$

The surface is generated by straight lines meeting the positive part of the axis of ξ, all of them being parallel to the plane $\xi = 0$.

As η/ξ tends to $-\infty$, ξ tends to $+0$.

The straight line $\xi = 0$, $\zeta = 0$ is a sharp edge of the surface.

Let us examine the section of the surface by the plane $\eta = b$, where $b > 0$, and see what happens as ξ tends to -0.

For points on the section

$$\xi = e^{b/\xi}, \quad d\xi/d\xi = -b\xi^{-2}e^{b/\xi}.$$

As ξ tends to -0, $d\xi/d\xi$ tends to -0.

Hence as a point on the section approaches the straight edge $\xi = 0$, $\zeta = 0$, the tangent to the section tends to coincide with the line in which $\eta = b$ meets $\xi = 0$. Hence as a point on the surface approaches the straight edge $\xi = 0$, $\zeta = 0$, the tangent plane tends to coincide with the plane $\xi = 0$, *i.e.* $z-x-y = 0$.

This accounts for the appearance of $z-x-y = 0$ as a special integral of the first class.

EXAMPLE 11.

$$(xe^{-2/\zeta}-ye^{-1/\zeta})(p-1)+(ye^{-2/\zeta}-xe^{-1/\zeta})(q-1) = \zeta^2(1-e^{-2/\zeta}),$$

where
$$\zeta = z-x-y.$$

The ordinary integrals are

$$u \equiv x+ye^{-1/\zeta} = \text{const.}, \quad v = y+xe^{-1/\zeta} = \text{const.}$$

The Lagrangian equations are

$$\begin{aligned} dx/(xe^{-2/\zeta}-ye^{-1/\zeta}) &= dy/(ye^{-2/\zeta}-xe^{-1/\zeta}) \\ &= dz/\{(x+y)(e^{-2/\zeta}-e^{-1/\zeta})+\zeta^2(1-e^{-2/\zeta})\} \\ &= d\zeta/\{\zeta^2(1-e^{-2/\zeta})\} = -d(1/\zeta)/(1-e^{-2/\zeta}). \end{aligned}$$

The last two Lagrangian ratios show that $\zeta = +0$ and $\zeta = \infty$ are special integrals.

In this case $J = 1-e^{-2/\zeta}$, and therefore tends to 1 as ζ tends to $+0$. Also $M = \zeta^2$.

It should be observed that $\zeta = +0$ makes the coefficients P, Q, R each equal to zero, and so constitutes an integral such as is recognised by Goursat and Chrystal. Moreover, since it makes $J = 1$, it is a special integral of the first class.

If we multiply right through by $e^{2/\zeta}$ and take the equation in the form

$$(x-ye^{1/\zeta})(p-1)+(y-xe^{1/\zeta})(q-1) = \zeta^2(e^{2/\zeta}-1),$$

then the Lagrangian equations are

$$dx/(x-ye^{1/\zeta}) = dy/(y-xe^{1/\zeta}) = dz/\{(x+y)(1-e^{1/\zeta})+\zeta^2(e^{2/\zeta}-1)\}$$
$$= d\zeta/\{\zeta^2(e^{2/\zeta}-1)\}.$$

The last Lagrangian ratio shows that $\zeta = -0$ is a special integral; but it is of the second class, as we see by putting u in the form

$$\{xe^{1/\zeta}+y\}/e^{1/\zeta} = \text{const.}$$

If we now apply § 8, we get as special integrals of the second class

$$xe^{1/\zeta}+y = 0 \quad \text{and} \quad e^{1/\zeta} = 0.$$

The latter requires $\zeta = -0$. This result is confirmed by calculating the value of J after taking, instead of u and v, their reciprocals as required by the understanding with regard to the calculation of J. J is then

$$(1-e^{-2/\zeta})(x+ye^{-1/\zeta})^{-2}(y+xe^{-1/\zeta})^{-2},$$

and this vanishes when ζ tends to -0.

But $\zeta = 0$ is not a true envelope.

To see this take a complete primitive of the equation

$$x+ye^{-1/\zeta} = a(y+xe^{-1/\zeta})+b,$$

i.e.,
$$(x-ay-b) = (ax-y)e^{-1/\zeta}.$$

If this surface be cut by a plane $\zeta = c$, the section is the straight line in which the plane $x-ay-b = (ax-y)e^{-1/c}$ meets the plane $\zeta = c$; *i.e.*, $\zeta = c$ meets the surface in a single straight line.

But if we make c tend to $+0$, the straight line is the intersection of $\zeta = 0$ and $x-ay-b = 0$.

Moreover the tangent plane to the surface at a point approaching the line $\zeta = 0$, $x-ay-b = 0$ tends to coincide with $x-ay-b = 0$.

On the other hand, if we make c tend to -0, the straight line is the intersection of $\zeta = 0$ and $ax-y = 0$.

Moreover, the tangent plane to the surface at a point approaching the line $\xi = 0,\ ax-y = 0$ tends to coincide with $ax-y = 0$.

Thus whilst the plane $\xi = 0$ meets the surface in the straight lines in which it meets the planes $x-ay-b = 0$ and $ax-y = 0$, it is not a true tangent plane. Each of these two straight lines is a sharp edge on the surface.

Although $\xi = 0$ is a solution of the equation as first written, such as Goursat and Chrystal recognise, it may be observed that if the equation be divided right through by ξ^2, and ξ be made to approach $+0$, then the equation is *not* satisfied.

EXAMPLE 12.

$$\{x\sin(\xi^r)-y\}(p-1)+\{y\sin(\xi^r)-x\}(q-1) = \frac{1}{r}\,\xi^{1-r}\cos(\xi^r),$$

where $r<1$ and $\xi = z-x-y$.

The ordinary integrals are

$$u \equiv x+y\sin(\xi^r) = \text{const.}, \qquad v \equiv y+x\sin(\xi^r) = \text{const.}$$

The Lagrangian equations are

$$dx/\{x\sin(\xi^r)-y\} = dy/\{y\sin(\xi^r)-x\}$$

$$= dz\Big/\Big[(x+y)\{\sin(\xi^r)-1\}+\frac{1}{r}\,\xi^{1-r}\cos(\xi^r)\Big]$$

$$= d\xi\Big/\Big[\frac{1}{r}\,\xi^{1-r}\cos(\xi^r)\Big].$$

The last Lagrangian ratio is indeterminate

(i) if $\xi = 0$ (since $r<1$),

(ii) if $\xi^r = \frac{1}{2}(2n+1)\pi$, where n is integral.

Now $$J = \cos^2(\xi^r), \qquad M = \frac{1}{r}\,\xi^{1-r}\sec(\xi^r).$$

Hence, if $\xi^r = \frac{1}{2}(2n+1)\pi$, J vanishes; and therefore this gives special integrals of the second class, which are all obtainable from the integrals

$$u \pm v = 0.$$

Consider next the solution $\xi = 0$.

If $r>0$ (that is to say, since $r<1$, if r be a positive proper fraction), then $J = 1$ when $\xi = 0$, and $\xi = 0$ is a special integral of the first class.

If $r < 0$, then when $\xi = 0$ both u and v are undetermined, and so also is J. But $\xi = 0$ makes $p = q = 1$, and the differential equation is satisfied.

It would appear therefore still to be a special integral of the first class ; but the point is a difficult one.

VII.

Third Group of Examples (*Nos.* 13–16).

These special integrals of the second class are illustrations of § 6. They arise when an ordinary integral $u = \text{const.}$, takes the form

$$st = \text{const.},$$

where $$s = \text{const.}, \quad t = \text{const.},$$

are not also ordinary integrals.

J vanishes, M may or may not vanish, on the special integral.

EXAMPLE 13.*

$$xp + yq = z.$$

Take $$u \equiv z/x, \quad v \equiv z/y.$$

Take then the integral $$u - \frac{v}{u} = 0,$$

i.e., $$\frac{z}{x} - \frac{x}{y} = 0.$$

This leads to the special integral given by Forsyth

$$z = \frac{x^2}{y}.$$

The Jacobian of $z/x, z/y, z - \frac{x^2}{y}$ is $\frac{z}{x^2y^2}\left(z - \frac{x^2}{y}\right)$, and vanishes on the special integral, whilst $M = x^2y^2/z$ does not vanish on the special integral.

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* Chrystal, *l.c.* ; Forsyth, *Theory of Differential Equations*, Vol. 5, p. 68.

EXAMPLE 14.*

$$x^2p-xyq+y^2=0.$$

The ordinary integrals are

$$u\equiv xy=\text{const.},\qquad v\equiv\frac{3zx-y^2}{3x^3y^2}=\text{const.}$$

Taking the value of the constant in v equal to zero, we have the special integral of the second class

$$3zx-y^2=0.$$

EXAMPLE 15.

$$y^2(z-x-y)^{\frac{1}{2}}p+y(y-2x)q$$
$$=y^2-2xy+4(x-y)(z-x-y)+y^2(z-x-y)^{\frac{1}{2}}.$$

The ordinary integrals are

$$u\equiv x+y(z-x-y)^{\frac{1}{2}},\qquad v\equiv x^2+y^2(z-x-y)^{\frac{1}{2}},$$
$$J=2(y-x)(z-x-y)^{\frac{1}{2}},\qquad M=-2(z-x-y)^{\frac{1}{2}}.$$

Thus both J and M vanish on the special integral (compare Ex. 25).

The Lagrangian equations are

$$\frac{dx}{y^2(z-x-y)^{\frac{1}{2}}}=\frac{dy}{y^2-2xy}=\frac{dz}{y^2-2xy+4(x-y)(z-x-y)+y^2(z-x-y)^{\frac{1}{2}}}$$
$$=\frac{d(z-x-y)}{4(x-y)(z-x-y)}.$$

This shows that $z-x-y=0$ is a special integral, and the form of J shows that it is of the second class.

It will be noted that the lowest exponent of $z-x-y$ in the coefficients is $\frac{1}{2}$.

This illustrates the point that it is not a *sufficient* condition for the existence of a special integral of the first class that the lowest exponent of the irrationality in the coefficients should be between 0 and 1 (see § 5).

If we form the integral $v-u^2=0$,

* Steinley, *American Journal of Mathematics*, Vol. 37 (1916), pp. 359–366. This paper contains very little in common with my work. It contains, however, this example and two others given below, Nos. 23 and 24, which I have examined as illustrations of my §§ 6 and 8.

$(z-x-y)^{\frac{1}{3}}$ is a factor of the left-hand side, thus verifying the statement that

$$z-x-y=0,$$

is a special integral of the second class.

This integral, however, possesses the envelope property like the special integrals of the first class.

EXAMPLE 16.*

$$(x^2+2xy)\,p-z^2q=y^2.$$

The Lagrangian equations are

$$dx/(x^2+2xy)=-dy/z^2=dz/y^2=d(x+y+z)/\{(x+y+z)(x+y-z)\}.$$

The last ratio becomes indeterminate when $x+y+z=0$, which therefore gives a special integral.

The ordinary integrals are

$$u\equiv y^3+z^3=a,$$

$$v=x^{-1}E-\int(a-y^3)^{-\frac{2}{3}}E\,dy=\text{const.},$$

where

$$E=\exp\left[-\int 2y(a-y^3)^{-\frac{2}{3}}\,dy\right],$$

and where, after the integrations have been performed, a is to be replaced by y^3+z^3.

Now it may be verified that

$$\int(a-y^3)^{-\frac{2}{3}}E\,dy=-E/[y+(a-y^3)^{\frac{1}{3}}],$$

and therefore

$$v=E\left\{x^{-1}+\left(1/[y+(a-y^3)^{\frac{1}{3}}]\right)\right\}=E\left(x^{-1}+(y+z)^{-1}\right)$$

$$=E(x+y+z)/\{x(y+z)\}=\text{const.}$$

This shows that $x+y+z=0$ and $y+z=0$ are special integrals of the second class (§ 8).

Moreover, since the factors of y^3+z^3 are $y+z$, $y+\omega z$, $y+\omega^2 z$, where $\omega^3=1$, it follows that $y+\omega z=0$ and $y+\omega^2 z=0$ are also special integrals of the second class (§ 6).

The Jacobian of u, v, $x+y+z$, is

$$3x^{-2}E(x+y+z)(x+y-z),$$

which vanishes on the special integral $x+y+z=0$, whilst $M=x^2/(3E)$.

In this example $PJ_x+QJ_y+RJ_z=-6Ex^{-2}y^2(x+y+z)$.

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* Forsyth, *l.c.*, Vol. 5, p. 72.

VIII.

Fourth Group of Examples (*Nos.* 17–20).

These special integrals of the second class are illustrations of § 8. They arise when an ordinary integral takes the form

$$s/t = \text{const.},$$

where $s = \text{const.}$, $t = \text{const.}$, are not also ordinary integrals.

If the Jacobian be calculated in accordance with the directions given in § 8, it vanishes on the special integral.

Example 17.

Consider the equation

$$(x-3y-3z)\,p-2(x-y+z)\,q = x+y+5z.$$

The ordinary integrals are

$$u \equiv x+y+z = \text{const.}, \qquad v \equiv (x-y+z)/(x-2y-z) = \text{const.}$$

Consider the special integral

$$x-y+z = 0.$$

In this case $M = (x-2y-z)^2,$

$$J = 4\,(x-y+z)(x-2y-z)^{-2}.$$

Thus M is finite on the special integral, but J vanishes there.

If we take for v the form $(x-2y-z)/(x-y+z)$, then

$$M = -\,(x-y+z)^2,$$

which vanishes on the special integral, but

$$J = -\,4\,(x-y+z)^{-1},$$

and is therefore infinite on the special integral. This form of v is, as explained in § 8, unsuitable for the examination of this special integral.

Example 18.

$$\text{(i)}\ p+q = z^2, \qquad \text{(ii)}\ p+q = z^n,$$

where n is a positive integer greater than 2.

(i) The ordinary integrals are

$$u \equiv y-x = \text{const.}, \qquad v \equiv (xz+1)/z = \text{const.}$$

The latter leads to the special integrals

$$z = 0 \quad \text{and} \quad xz+1 = 0.$$

The Jacobian of $y-x$, $z/(xz+1)$, and z is $z^2/(xz+1)^2$, and M is $(xz+1)^2$.

The Jacobian of $y-x$, $(xz+1)/z$, and $xz+1$ is $-(xz+1)/z$, but M is $-z^2$.

Each Jacobian vanishes on its own special integral, but M does not.

There is, however, a point to be noticed in this case in connection with the Lagrangian system of equations,

$$dx/1 = dy/1 = dz/z^2.$$

The only solution given by Cauchy's theorem for the expansion of z in powers of x, if z is to have the value $z = 0$ when $x = 0$, is $z = 0$. This and the following example are cases of the application of Cauchy's theorem, which are regarded as exceptional on pp. 31, 32, Vol. 2, of Forsyth's *Theory of Differential Equations.*

(ii) In the case of the equation $p+q = z^n$, where n is a positive integer greater than 2, the Lagrangian equations are

$$dx/1 = dy/1 = dz/z^n,$$

and here again the only solution given by Cauchy's theorem for the expansion of z in powers of x, if z is to have the value $z = 0$ when $x = 0$, is $z = 0$.

EXAMPLE 19.*

$$xp-yq = (z-xy)^2.$$

The ordinary integrals are

$$u \equiv xy = \text{const.}, \qquad v \equiv (z-xy)/[(z-xy)\log x+1] = \text{const.}$$

Forsyth gives the special integral

$$z-xy = 0.$$

It follows also from § 8 that

$$(z-xy)\log x+1 = 0$$

is another special integral.

* Forsyth, *Differential Equations*, p. 383.

The Jacobian of

$$xy, \quad (z-xy)/[(z-xy)\log x+1], \quad z-xy,$$

is $$(z-xy)^2/[(z-xy)\log x+1]^2,$$

and M is $$[(z-xy)\log x+1]^2.$$

The Jacobian of

$$xy, \quad [(z-xy)\log x+1]/(z-xy), \quad (z-xy)\log x+1,$$

is $$-[(z-xy)\log x+1]/(z-xy),$$

and M is $$-(z-xy)^2.$$

EXAMPLE 20.*

$$xp+2yq = 2[z-(x^2/y)]^2.$$

The ordinary integrals are

$$u \equiv (x^2/y) = \text{const.} \qquad v \equiv [z-(x^2/y)]/[\{z-(x^2/y)\}\log y+1] = \text{const.}$$

Forsyth gives the special integral

$$z-(x^2/y) = 0.$$

Another special integral is derivable from the form of v by § 8, viz.,

$$[z-(x^2/y)]\log y+1 = 0.$$

The Jacobian of

$$x^2/y, \quad \{z-(x^2/y)\}/[\{z-(x^2/y)\}\log y+1], \quad z-(x^2/y),$$

is $$-2xy^{-2}[z-(x^2/y)]^2/[\{z-(x^2/y)\}\log y+1]^2,$$

and M is $$-y^2x^{-1}[\{z-(x^2/y)\}\log y+1]^2.$$

The Jacobian of

$$(x^2/y), \quad [\{z-(x^2/y)\}\log y+1]/\{z-(x^2/y)\}$$

and $$[\{z-(x^2/y)\}\log y+1]$$

is $$2x[\{z-(x^2/y)\}\log y+1]/[y^2\{z-(x^2/y)\}],$$

and M is $$y^2x^{-1}\{z-(x^2/y)\}^2.$$

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* Forsyth, *Theory of Differential Equations*, Vol. 5, p. 69.

IX.

Fifth Group of Examples (*Nos.* 21, 22).

These examples are illustrations of § 12. The coefficients of the differential equations are regular for all finite values of the variables. The Lagrangian system of equations are such that the only solutions given by the application of Cauchy's theorem assign constant values to some of the dependent variables. The special integrals which arise are all of the second class.

EXAMPLE 21.

$$[a(z-x-y)+b](p-1)+[c(z-x-y)+k](q-1) = z-x-y.$$

The Lagrangian equations are

$$\begin{aligned} dx/[a(z-x-y)+b] &= dy/[c(z-x-y)+k] \\ &= dz/[(a+c+1)(z-x-y)+b+k] \\ &= d(z-x-y)/(z-x-y). \end{aligned}$$

Hence $z-x-y=0$ is a special integral.

The ordinary integrals are

$$u \equiv (bc-ak)(z-x-y)+kx-by = \text{const.},$$

$$v \equiv (z-x-y)\exp\left[\frac{a}{b}(z-x-y)-\frac{x}{b}\right] = \text{const.}$$

Then $$J=-v, \quad M=-\exp\left[\frac{x}{b}-\frac{a}{b}(z-x-y)\right].$$

If v had been taken in the form

$$v = \frac{a}{b}(z-x-y)-\frac{x}{b}+\log(z-x-y),$$

it would have appeared that $J=-1$, and the special integral would have seemed to belong to the first class, whereas it really belongs to the second class.

If we put $w=z-x-y$, the Lagrangian equations become

$$dy/dx = (cw+k)/(aw+b), \qquad dw/dx = w/(aw+b).$$

Now the only integral furnished by Cauchy's theorem for the last equation, if w is to vanish when x vanishes, is $w = 0$, which, with the preceding equation, gives

$$w = 0, \qquad y = (kx/b) + \text{const.},$$

as a system of integrals of the Lagrangian equations.

EXAMPLE 22.

$$[a(z-x-y)^2+b](p-1)+[c(z-x-y)^2+k](q-1) = (z-x-y)^2.$$

The Lagrangian equations are

$$\begin{aligned} dx/[a(z-x-y)^2+b] &= dy/[c(z-x-y)^2+k] \\ &= dz/[(a+c+1)(z-x-y)^2+b+k] \\ &= d(z-x-y)/(z-x-y)^2. \end{aligned}$$

The ordinary integrals are

$$u \equiv kx-by+(bc-ak)(z-x-y) = \text{const.},$$

$$v \equiv (z-x-y)/[a(z-x-y)^2-x(z-x-y)-b] = \text{const.}$$

Applying § 8 tc the form of v, it is seen that

$$z-x-y = 0 \quad \text{and} \quad a(z-x-y)^2-x(z-x-y)-b = 0$$

are special integrals of the second class.

The Jacobian of u, v, $z-x-y$ is bv^2, and

$$M = [a(z-x-y)^2-x(z-x-y)-b]^2/b.$$

Putting $z-x-y = w$, the Lagrangian equations are equivalent to

$$dy/dx = (cw^2+k)/(aw^2+b), \qquad dw/dx = w^2/(aw^2+b),$$

and the only solution of the second of these equations given by Cauchy's theorem, if w is to vanish when x vanishes, is $w = 0$. This furnishes the special integral and reduces the first equation to

$$dy/dx = k/b,$$

and therefore

$$y = (kx/b)+c.$$

X.

Sixth Group of Examples.

Differential Equations with Transcendental Coefficients.

EXAMPLE 23.*

$$(z-e^y)\,p+q=z.$$

The ordinary integrals are

$$u \equiv x-z+e^y = \text{const.}, \qquad v \equiv e^y/z = \text{const.}$$

The latter integral can be written

$$1-e^y/z = \text{const.},$$

i.e.,

$$(z-e^y)/z = \text{const.}$$

Hence, by § 8, both $z-e^y=0$ and $z=0$ are special integrals of the second class.

EXAMPLE 24.*

$$p\tan x+q\tan y=\tan z.$$

The ordinary integrals are

$$u \equiv \sin z/\sin x = \text{const.}, \qquad v \equiv \sin z/\sin y = \text{const.}$$

Form the ordinary integral

$$u^{n+1}-v^n=0,$$

i.e.,

$$\sin^n z\,\{\sin z-(\sin^{n+1} x/\sin^n y)\}/\sin^{n+1} x = 0.$$

(i) The factor $\sin z-(\sin^{n+1} x/\sin^n y)=0$ gives Steinley's special integral.

(ii) Taking the other factor, we have

$$\sin z = 0,$$

and therefore $z=n\pi$ is a special integral of the second class.

* Steinley, *l.c.*

(iii) Taking the ordinary integral $u = 1$, we have

$$\sin z = \sin x,$$

and therefore $z = n\pi + (-1)^n x.$

Taking $u = -1$, we get $z = n\pi + (-1)^n (-x).$

These two together give $z = n\pi \pm x.$

(iv) Similarly, $z = n\pi \pm y.$

All these are special integrals of the second class.

EXAMPLE 25.*

$$[1+(z-x-y)\log(z-x-y)]p+q = 2.$$

The Lagrangian equations are

$$\begin{aligned} dx/[1+(z-x-y)\log(z-x-y)] &= dy/1 = dz/2 \\ &= -d(z-x-y)/\{(z-x-y)\log(z-x-y)\} \\ &= -d(z-x-y-1)/\{(z-x-y)\log(z-x-y)\}, \end{aligned}$$

showing that $z-x-y = 0$ and $z-x-y = 1$

are special integrals.

The ordinary integrals are

$$z-2y = \text{const.}, \quad e^y \log(z-x-y) = \text{const.}$$

Treating the second ordinary integral as an example of § 6, we make

$$\log(z-x-y) = 0.$$

This gives the special integral of the second class,

$$z-x-y = 1.$$

The Jacobian of $z-2y$, $e^y \log(z-x-y)$, $z-x-y-1$, is

$$e^y \log(z-x-y),$$

which vanishes on the special integral, and

$$M = -e^{-y}(z-x-y),$$

which does not vanish there.

* Recently communicated to me by Prof. Forsyth.

If, however, we write the second ordinary integral

$$e^{-y} \times [1/\log(z-x-y)] = \text{const.},$$

and then make the factor $1/\log(z-x-y)$ vanish, we get the special integral

$$z-x-y = 0.$$

In this case the Jacobian of $z-2y$, $e^{-y}/\log(z-x-y)$ and $z-x-y$ is

$$-e^{-y}/\log(z-x-y),$$

whilst M is $\qquad e^{y}(z-x-y)[\log(z-x-y)]^2.$

In this case both J and M vanish on the special integral, as in the case of Ex. 15. This possibly marks some difference in character between the special integral $z-x-y=0$, and the one first considered, $z-x-y=1$, in which the Jacobian vanishes, but M does not vanish, on the special integral.

EXAMPLE 26.

$$(1+\sin m\xi)\, p+q = 2, \quad \text{where} \quad \xi = z-x-y.$$

The Lagrangian equations are

$$dx/(1+\sin m\xi) = dy/1 = dz/2 = -d\left(\xi - \frac{s\pi}{m}\right)\Big/\sin m\xi.$$

Hence $\qquad \xi - \frac{s\pi}{m} = 0,$

where s is zero or any integer, is a special integral.

The ordinary integrals are

$$u \equiv z-2y = \text{const.}, \qquad v \equiv e^{my}\sin \tfrac{1}{2}m\xi/\cos \tfrac{1}{2}m\xi = \text{const.}$$

Applying the method of § 8 to v, we get

(i) By making $\sin \frac{1}{2}m\xi = 0$, the special integral $\xi = 2r\pi/m$, where r is zero or any positive or negative integer. In this case $J = me^{my}\tan \frac{1}{2}m\xi$.

(ii) By making $\cos \frac{1}{2}m\xi = 0$, the special integral $\xi = (2r+1)\pi/m$. To get the corresponding J it is necessary to take $v = e^{-my}\cot \frac{1}{2}m\xi$, and then $J = -me^{-my}\cot \frac{1}{2}m\xi$.

Putting the results together, it is seen that the special integrals are given by $\xi = s\pi/m$, where s is integral, as has been already shown by the Lagrangian equations.

EXAMPLE 27.*

$$[1+(\sin m\xi)\log\xi]p+q=2, \quad \text{where} \quad \xi=z-x-y.$$

The Lagrangian equations are

$$dx/[1+(\sin m\xi)\log\xi]=dy/1=dz/2=-d\left(\xi-\frac{s\pi}{m}\right)\Big/[(\sin m\xi)\log\xi],$$

showing that $\xi=s\pi/m$ is a special integral. The ordinary integrals are

$$z-2y=\text{const.}, \quad y+\int[d\xi/\{(\sin m\xi)\log\xi\}]=\text{const.}$$

But it will now be shown that the form of the second integral has not been taken in accordance with the understanding for the calculation of J.

I shall examine first the special integral $\xi=0$. To do this it is necessary to separate out the part of the integral

$$\int[d\xi/\{(\sin m\xi)\log\xi\}]$$

which becomes infinite when $\xi=0$. Now

$$\int[d\xi/\{(\sin m\xi)\log\xi\}]$$

$$=\int\frac{d\xi}{\log\xi}[(m\xi)^{-1}+\{\operatorname{cosec} m\xi-(m\xi)^{-1}\}]$$

$$=m^{-1}\log(\log\xi)+\int[m\xi-\sin m\xi)/\{(m\xi\sin m\xi)\log\xi\}]\,d\xi.$$

Now $(m\xi-\sin m\xi)/\{(m\xi\sin m\xi)\log\xi\}$ is continuous to the right of 0; hence if it be integrated from 0 to a positive upper limit the result is convergent. Representing its value by $E(\xi)$ we may put

$$\int_0 d\xi/[(\sin m\xi)\log\xi]=m^{-1}\log(\log\xi)+E(\xi),$$

where $E(\xi)$ remains finite when ξ tends to $+0$.

Hence we may put the second ordinary integral into the form

$$y+m^{-1}\log(\log\xi)+E(\xi)=\text{const.},$$

* Recently communicated to me by Prof. Forsyth.

or into the form

$$(\log \xi) \exp[my+mE(\xi)] = \text{const.},$$

where $E(\xi)$ remains finite when ξ tends to $+0$.

Hence, arguing as in Ex. 25, we see that $\xi = 0$, *i.e.* $z-x-y = 0$, is a special integral of the second class.

Consider next the special integrals given by $m\xi = s\pi$, where s is integral but not zero.

Put $\xi = \frac{s\pi}{m} + \epsilon$, then the second ordinary integral

$$v = y+(-1)^s \int d\epsilon \Big/ \left[(\sin m\epsilon) \log \left(\frac{s\pi}{m} + \epsilon \right) \right] = \text{const.}$$

Now replacing $1 \Big/ \log \left(\frac{s\pi}{m} + \epsilon \right)$ by

$$\left[1 \Big/ \log \frac{s\pi}{m} \right] - \left[\left\{ \log \left(1 + \frac{m\epsilon}{s\pi} \right) \right\} \Big/ \left\{ \log \frac{s\pi}{m} \log \left(\frac{s\pi}{m} + \epsilon \right) \right\} \right],$$

the integral appearing in the value of v is

$$\left[1 \Big/ \log \frac{s\pi}{m} \right] \int d\epsilon / \sin m\epsilon$$

$$- \left[1 \Big/ \log \frac{s\pi}{m} \right] \int d\epsilon \left[\log \left(1 + \frac{m\epsilon}{s\pi} \right) \Big/ \left\{ \log \left(\frac{s\pi}{m} + \epsilon \right) \sin m\epsilon \right\} \right],$$

and it is sufficient to show that the second integral in the expression last written has a finite limit when ϵ tends to 0.

Now $\log \left(1 + \frac{m\epsilon}{s\pi} \right)$ has a finite ratio to $m\epsilon$, and $\log \left(\frac{s\pi}{m} + \epsilon \right)$ has a finite ratio to $\log \frac{s\pi}{m}$

Hence the integral has a finite ratio to

$$\int \{ m\epsilon \, d\epsilon / \sin m\epsilon \}.$$

But if ϵ be small enough, $\quad m\epsilon / \sin m\epsilon < \frac{1}{2}\pi.$

Hence this integral is finite.

Hence the ordinary integral v is

$$E(\epsilon)+y+\left[(-1)^s\Big/\log\frac{s\pi}{m}\right]\int\{d\epsilon/\sin m\epsilon\}=\text{const.},$$

where $E(\epsilon)$ has a finite value when ϵ tends to zero; *i.e.* it is

$$E\left(\xi-\frac{s\pi}{m}\right)+y+\left[(-1)^s\Big/\log\frac{s\pi}{m}\right]\log\tan\tfrac{1}{2}(m\xi-s\pi)=\text{const.},$$

or

$$[\sin\tfrac{1}{2}(m\xi-s\pi)/\cos\tfrac{1}{2}(m\xi-s\pi)]\exp\left[(-1)^s\log\frac{s\pi}{m}\left\{y+E\left(\xi-\frac{s\pi}{m}\right)\right\}\right]=\text{const.}$$

Hence by §§ 6 and 8, it follows that

$$\xi-\frac{s\pi}{m}=0$$

is a special integral of the second class.

If we had taken v in the last form but one given for it, the value for J would have appeared to be 1, and it would have seemed that the integral was of the first class.

If we take for v the form

$$y+\int[d\xi/\{\sin m\xi\,.\log\xi\}],$$

it is infinite on the special integral $\xi=0$, and we must replace it by $\exp(e^v)$. Let us call this v', then, taking v' for the second ordinary integral instead of v in the calculation of J, $J=v'\log v'$, and so vanishes when v' vanishes.

EXAMPLE 28.

$$[1+\text{arc}\tan\xi]\,p+q=2,\quad\text{where}\quad\xi=z-x-y.$$

The Lagrangian equations are

$$dx/[1+\text{arc}\tan\xi]=dy/1=dz/2=-d\xi/\text{arc}\tan\xi,$$

showing that $\xi=0$ or $s\pi$, where s is any integer, is a special integral.

The ordinary integrals are

$$u \equiv z-2y = \text{const.}, \qquad v \equiv y+\int d\zeta/\arctan\zeta = \text{const.}$$

Putting $\zeta = \tan\phi$,

$$\int d\zeta/\arctan\zeta = \int \sec^2\phi\, d\phi/\phi$$
$$= \int d\phi/\phi + \int \tan^2\phi\, d\phi/\phi = \log\phi + E(\phi),$$

where $E(\phi)$ tends to a finite value as ϕ tends to 0.

Hence the second ordinary integral is

$$y+\log\phi+E(\phi) = \text{const.},$$

or

$$\phi \exp[y+E(\phi)] = \text{const.}$$

Applying § 6, we have a special integral of the second class given by $\phi = 0$, which leads to $z-x-y = 0$.

In like manner $z-x-y = s\pi$, where s is any integer, is a special integral of the second class.

XI.

Seventh Group of Examples (*Nos.* 29–30). *Illustrations of* §§ 3 *and* 12 (B).

EXAMPLE 29.

Consider the equation

$$[(z+ax+by)^{\frac{1}{2}}(b-l)-b+g]\,p+[(z+ax+by)^{\frac{1}{2}}(k-a)-e+a]\,q$$
$$-[(z+ax+by)^{\frac{1}{2}}(al-bk)-ag+be] = 0,$$

where a, b, e, g, k, l are constants.

The ordinary integrals are

$$u \equiv z+ax+by = C,$$
$$v \equiv (z+kx+ly)(z+ax+by)^{\frac{1}{2}}-(z+ex+gy) = C'.$$

To find the focal points we combine the equations

$$z+ax+by = C,$$
$$(z+ax+by)(z+kx+ly)^2 = (z+ex+gy+C')^2,$$

and $$z+ex+gy+C'=0.$$

These reduce to

$$z+ax+by=C, \quad z+ex+gy=-C', \quad z+kx+ly=0,$$

showing that all the focal points lie on the plane

$$z+kx+ly=0.$$

Now the cubic surface

$$(z+ax+by)(z+kx+ly)^2=(z+ex+gy+C')^2$$

touches the plane $$z+ax+by=0,$$

along the line in which it meets the plane

$$z+ex+gy+C'=0.$$

Hence, since C' is a variable parameter,

$$z+ax+by=0$$

is the locus of the lines of contact of the cubic surfaces obtained by varying C'.

It is therefore an envelope, and it satisfies the differential equation.

But the cubic surface has a double line whose equation is

$$z+kx+ly=0, \quad z+ex+gy+C'=0.$$

The locus of these double lines is therefore

$$z+kx+ly=0,$$

which here represents the focal surface.

But $z+kx+ly=0$ is not a proper envelope. Moreover it is not an integral of the differential equation at all.

In this case $M=1$, $J\equiv 0$.

All points on the plane $z+ax+by=0$ are critical points of the coefficients, and also of the ordinary integrals $v=$ const.

But the locus $z+ax+by=0$ is not a special integral.

It is a particular form of the ordinary integral

$$z+ax+by=C,$$

obtained by taking $$C=0.$$

It is to be noted that it is *not* an *isolated* portion of a particular form of the ordinary integral.

EXAMPLE 30.

$$(z-x-y)\log(z-x-y)[a(p-1)+b(q-1)]+k(p-1)+l(q-1)=0,$$

where a, b, k, l are constants.

The ordinary integrals are

$$u \equiv z-x-y = C, \qquad v \equiv e^{lx-ky}(z-x-y)^{[(z-x-y)(bx-ay)]} = C'.$$

In this case $M = v^{-1}$, and, taking $w = z-x-y$, J is *identically* zero.

All points on the plane $z-x-y=0$ are non-regular points of the coefficients, and also of the integrals v.

But the locus $z-x-y=0$ is not a special integral. It is a particular form of the ordinary integral $z-x-y=C$. It is *not* an *isolated* portion of a particular form of the ordinary integral.

ON MULTIPLE INTEGRATION BY PARTS AND THE SECOND THEOREM OF THE MEAN

By Prof. W. H. Young.

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1. In the present communication I propose to prove the general theorem of Integration by Parts in a form which takes account of recent progress in the Theory of Integration, and to deduce from it the Second Theorem of the Mean, in an extended form and for any number of variables.* As far as I am aware this mode of procedure differs from that adopted by earlier authors, and the results obtained present some novelty. As regards the method, this involves the theory of integration with respect to a function of bounded variation, and depends for its

* The account of the Formula for Integration by Parts and the Second Theorem of the Mean in Hobson's Treatise is concerned only with functions of a single variable, having Riemann integrals, and a paper by him in these *Proceedings*, 1909, Ser. 2, Vol. 7, p. 14, confines itself to extensions to Lebesgue integrals, again for the case of a single variable. The Theorem of the Mean for two variables for the integral $\int fg\,dx\,dy$ had been already given in 1903 by M. Krause (*Leipziger Bericht, math. Phys. Klasse*, pp. 239–263, "Mittelwertsätze in Gebiete d. Doppelsummen u. Doppelintegrale"). Krause supposes f bounded and integrable (in Riemann's sense, and g a function whose simple and double increments are of constant sign. He also, as he says, for simplicity of exposition, supposes g to be continuous. In 1905, in a paper which I cited on a previous occasion (see "Multiple Fourier Series," *Proc. London Math. Soc.*, Ser. 2, Vol. 11, p. 138, 1912), G. H. Hardy considers the same integral for the case in which g is any function of bounded variation, but f is subject to the same restrictions as in the work of Krause. A paper by Lebesgue which appeared in August, 1910, in the *Annales de l'Ecole Normale*, "Sur l'intégration des fonctions discontinues," contains a form of the Second Theorem of the Mean for functions of any number of variables, which I have had occasion to use in a previous communication to the Society. Hardy gives no reference to Krause, and Lebesgue mentions neither Krause nor Hardy. The reference in my paper on "Multiple Fourier Series" is to an earlier paper by Krause, and my acquaintance with the one here referred to is an indirect consequence of a reminder by a kindly referee of those earlier references. In the work of none of these authors, and indeed, as far as I know, of none, since Weierstrass, is the Formula for Integration by Parts utilised in obtaining the Theorem of the Mean. As is well known, Weierstrass confined himself to a single variable, and to a statement of the Theorem of the Mean involving considerable restrictions on the nature of the functions. Cf. Hobson, *loc. cit.*, p. 363.

generality on a property of these functions recently brought to light, which in the case of two variables is that the discontinuities of the function lie on a countable set of lines parallel to the two axes.*

As is well-known non-absolutely convergent integrals, when the functions considered involve more than one independent variable, have scarcely been considered with any sort of generality. An extension to the case of such integrals of the results of the paper is however briefly indicated.

Two main forms of the Formula for Integration by Parts are obtained. Taking for simplicity of statement the case of two variables, and supposing $f(x, y)$ and $g(x, y)$ both to have bounded variation, and one of them to be continuous, the first of these two forms is the following:—

$$\int_{0,\,0}^{a,\,b} g(x, y)\,df(x, y) = \Big[fg\Big]_{0,\,0}^{a,\,b} - \int_{x=0}^{a}\Big[f\,dg\Big]_{y=0}^{b} - \int_{y=0}^{b}\Big[f\,dg\Big]_{x=0}^{a} + \int_{0,\,0}^{a,\,b} f\,dg,$$

where, with the usual notation, $\Big[f\Big]_{y=0}^{b}$ means the increment of $f(x, y)$ with respect to y, viz., $f(x, b)-f(x, 0)$, and the same notation is used for the higher increments, so that

$$\Big[f\Big]_{0,\,0}^{a,\,b} = f(0, 0)-f(a, 0)-f(0, b)+f(a, b).$$

The second form is deduced from the first by an algebraic transformation, and has as many variants as the fundamental hyper-rectangle has corners. That in which the value of g at the origin explicitly occurs is for two variables the following:—

$$\int_{0,\,0}^{a,\,b} g(x, y)\,df(x, y) = g(0, 0)\Big[f\Big]_{0,\,0}^{a,\,b} + \int_{0,\,0}^{a,\,b}\Big[f\Big]_{x,\,y}^{a,\,b} dg(x, y) + \int_{0}^{a}\Big[f\Big]_{x,\,0}^{a,\,b} dg(x, 0) + \int_{0}^{b}\Big[f\Big]_{0,\,y}^{a,\,b} dg(0, y).$$

By supposing one or both of the functions $f(x, y)$, $g(x, y)$ to be integrals with respect to xy, or, more generally, with respect to any function of bounded variation, special, but very important, cases of these formulæ are obtained, in virtue of a Theorem of Substitution proved by me in a recent paper, and stated below as a Lemma (2 of § 4).†

* This may be proved by an argument similar to that adopted by my wife and myself in a forthcoming paper. It is, in fact, a particular case of a general theorem which forms one of the main objects of that paper, which deals with the discontinuities of the most general function of any number of variables.

† "On Multiple Integrals," *Proc. Roy. Soc.*, 1916, § 4, Equations 4 and 5.

The various forms of the Second Theorem of the Mean are deduced immediately from the second form of the Formula for Integration by Parts, and have as many variants as this formula. The value of g which occurs explicitly has reference to the corner opposite that which appears in the Formula for Integration by Parts to which it corresponds and from which it is deduced. Thus, if the function g and all its increments, taken with respect to $x_1, x_2, \ldots, x_n$, (x_1, x_2), $(x_1, x_3), \ldots, (x_{n-1}, x_n)$, (x_1, x_2, x_3), $\ldots, (x_1, x_2, \ldots, x_n)$, are all positive, the Theorem of the Mean states that

$$\int_0^a g\,df = g(a)\int_X^a df$$

[where the single letter or figure a, X, or 0, stands for the ensemble $(a_1, a_2, \ldots, a_n)$, $(X_1, X_2, \ldots, X_n)$, $(0, 0, \ldots, 0)$], X being a certain point in the hyper-rectangle $(0, a)$.

If, on the other hand, g is not known to be positive, but has all its increments positive, we have the second form, as follows :—

$$\int_0^a g\,df = g(0)\int_0^a df + [g(a) - g(0)]\int_X^a df.$$

More generally, if the increments of g taken with respect to r of the n variables singly be positive, and with respect to the remaining $(n-r)$ singly be negative, and if the double, triple, ..., n-ple increments have signs determined by the simple law that they are those which would be possessed by a function of n variables defined as the product of n functions, of which the first is a function of x_1 only, the second of x_2 only, and so on, these latter functions having their increments positive or negative in the manner assigned, formulæ precisely analogous to these hold good. They may either be obtained in precisely the same way, from the variants of the Formula for Integration by Parts, like those explicitly given above, or they may be deduced from these latter by a transformation of the independent variables of the form

$$x_r = a_r - y_r, \qquad x_s = y_s.$$

Evidently also, by supposing f to be an integral with respect to xy, or with respect to another function whose increments are all of the same sign, particular cases of great importance are obtained immediately.

It should also be noticed that various generalisations are possible. Thus, if the discontinuities of f and g are suitably arranged with respect to one another, the various formulæ described may still hold good when both the functions are discontinuous.

Moreover, as it is possible in a large and important class of cases to extend the Formula for Integration by Parts still further, *e.g.*, to non-absolutely convergent integration, we have *ipso facto*, as the reasoning adopted is purely formal, corresponding Theorems of the Mean.

In the same connection it will be remarked that the values $g(0)$ and and $g(a)$ of the function $g(x)$ may, when g is discontinuous, be replaced by the unique limits which g necessarily has at the points 0 and a, for approach to the point x in the open quadrant, or hyper-quadrant, under consideration.

The paper concludes by a modification in the direction of greater simplicity of a form of the Theorem of the Mean obtained by Lebesgue, and already referred to in a previous footnote. The formula is applicable when g is any function of bounded variation, and runs as follows :—

$$\left| \int_0^a f(x)\, g(x)\, dx - g(0) \int_0^a f(x)\, dx \right| \leqslant MV(a),$$

where V is the total variation of the function g, and M is the upper bound of $\int_x^a f\,dx$ in the hyper-rectangle $(0, a)$.*

2. It is of some importance to remark that, in basing the Theorem of the Mean on the formula for Integration by Parts, we have this advantage that it follows *ipso facto* that, if one of the functions of bounded variation f and g belong to any class of integrals, for which the formula for Integration by Parts is known to hold, or can be proved to hold, then the Theorem of the Mean in Bonnet's and allied forms is also true, provided only such integrals are continuous, or, more generally, assume all values between their upper and lower bounds. In this way we obtain the result that in a large and extremely important class of cases, the Theorem of the Mean is true for non-absolutely convergent multiple integrals.† In this connexion it is worthy of notice that the conditions under which the Theorem of the Mean has been shown to hold in the present communication, suggest that

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* This introduction, including the first footnote, has been rewritten since the presentation of the paper. It appeared to me, on re-reading it in the original form, that this did not sufficiently bring out the extent of the generality gained by the adoption of the methods of the paper. The change, however, is merely in the form and not in the substance.—[*January 9th*, 1917.]

† See a recent communication by myself to the Society, "On Integrals which are not necessarily Continuous."

the proper way to attack the problems presented when the step is taken in Analysis which consists in passing from a function of a single variable to a function of any number of variables, is to begin with the consideration of such functions of n variables as present the same peculiarities *mutatis mutandis* as those of the function got by multiplying together n functions each of a single variable. In particular, corresponding to functions of a single variable, possessing a Harnack-Lebesgue or Denjoy integral, or a Y-integral, with a set of Harnack points forming, for example, any perfect set of points dense nowhere, with a set of black intervals dense everywhere, we have non-absolutely convergent integrals, whose discontinuities lie on lines parallel to the coordinate axes, and form a set constituted by the external and semi-external points of a set of black rectangles dense everywhere. For the comparatively general class of functions whose discontinuities of the Harnack type are distributed in such a manner, and which at the same time possess an integral of the type in question, the Theorem of the Mean may in fact be shown to be true, as the reasoning I have employed on the straight line becomes then available for the plane. Moreover the corresponding statement is true for functions of any number of variables.

3. If the function be one of n variables, it will possess n single increments, $\frac{1}{2}n(n-1)$ double increments, and $(n!)/(r!\ n-r!)$ r-ple increments, for all values of r from 1 to n, so that there will be in all 2^n-1 increments of various dimensions. It is clear then that there are no less than

$$2^{2^n-1}$$

types of *monotone functions of n-variables*.* There are, however, only

$$2^n$$

types dealt with in the Theorem of the Mean as we have stated it. There is no difficulty in obtaining other variants of the theorem, less elegant in form. This may be left as an exercise to the reader; they are not found to be useful in practice. It is preferable to employ the general theorem of Integration by Parts.

It is of considerable importance to notice that, numerous as the types are, all of them are expressible as the algebraic sum of a number of func-

* Thus with five variables the number is over two hundred million. Of course if we count, as we should do, the types the same when they are got from one another by permutations of the variables, this number must be reduced.

tions, each of which has the simplest type of all, namely, that in which all the increments are positive; which, in two dimensions, we may call a *monotonely monotone increasing function.*

To see this we have only to use systematically the two facts—

(i) *that the r-ple increment of an s-ple increment is an (r+s)-ple increment;*

(ii) *that, if $r > n$, the r-ple increment of a function of n variables vanishes identically.*

Suppose the function to have positive n-ple increment with respect to its n variables $x_1, x_2, \ldots, x_n$, and that we are considering it with respect to the hyper-rectangle of which the first and last corners are respectively the origin and the point $a_1, a_2, \ldots, a_n$. Denote by $q(x_1, x_2, \ldots, x_n)$ the n-ple increment of $f(x_1, x_2, \ldots, x_n)$, taken over the rectangle the last of whose corners is $x_1, x_2, \ldots, x_n$ and whose opposite corner is the origin. This function q is a positive function of n variables having all its increments without exception positive.

In fact, consider the s-ple increment of q relating to the first s of the n variables. Only those terms of q which contain all these s variables will have a non-vanishing s-ple increment. But the terms involving these s variables constitute an $(n-s)$-ple increment of f taken with respect to the remaining $(n-s)$ variables, over the hyper-rectangle whose first and last corners are the origin and the points $(0, 0, \ldots, 0, x_{s+1}, x_{s+2}, \ldots, x_n)$. It is thus at once evident that the s-ple increment we are considering is the n-ple increment of f over the hyper-rectangle whose first and last corners are

$$(x_1, x_2, \ldots, x_s, 0, 0, \ldots, 0) \quad \text{and} \quad (x_1+h_1, x_2+h_2, \ldots, x_s+h_s, x_{s+1}, \ldots, x_n),$$

and is therefore positive.

If the n-ple increment of f is negative, we have merely to subtract the function from any convenient quantity, *e.g.*, one greater than its upper bound, to have a function whose n-ple increment is positive. To this new function we then apply the above process, and obtain a similar function q. In either case this function q is the sum or difference of the original function f and a number of functions of a smaller number of variables. Conversely f is thus expressed as the sum or difference of a function of n variables of the type required, and functions, not usually of this type, but containing a smaller number of variables. To these latter functions with a smaller number of variables we apply the same process. Continuing in

this way we may ultimately express the original function f as the algebraic sum, each with proper sign, of a number of functions, all of which are of the type desired. Evidently there will be in general, when functions of the same kind and involving the same variables are added together, twice as many functions as the total number of combinations of n things, taken 1, 2, 3, ..., n times, that is $2^{n+1}-2$.

It will be noticed that the types considered in the extension of Bonnet's theorem are the same as those of the simple functions obtained by multiplying together n monotone functions each of one only of the n variables. The signs of all the increments are the same as those of such a simple function, when the single increments have been assigned in any arbitrary manner.

Instead of reducing our monotone functions to functions all of whose increments are positive, we may reduce them to functions of any one of these special types. In the function denoted by the n-th increment of f over the rectangle whose near point is the origin, and whose further point is $(x_1, x_2, \ldots, x_n)$, we have merely to replace a suitable set of the x's by a's, and the corresponding zeros by x's, to consider, in fact, such a function as, for example,

$$\left[f\right]_{x_1, x_2, 0, x_4, 0, 0, \ldots, 0}^{a_1, a_2, x_3, a_4, x_5, \ldots, x_n}.$$

Here, if we suppose the n-ple increment of f positive, so that this new function is also positive, its increment with respect to x_1 or x_2 or x_4 is negative, and with respect to each of the other letters is positive. The double, triple, &c., increments will have the corresponding signs. There is no difficulty therefore in reducing our given function f to functions of this type.

These monotone functions are not the most general functions of bounded variation. It is clear, in fact, that only one of the two variations (with respect to all n variables) exists; the remaining one is zero.

4. As stated in the introduction the method employed is that of integration with respect to a function of bounded variation.* We need to

* For information on this subject I refer my readers to my earlier papers, as well as to a more recent one presented to the Royal Society. It may be remarked that, for example, *ordinary double integration is integration with respect to the simple function of bounded variation constituted by the product of the two variables x and y.*

borrow from this theory the following simple results, which it will be convenient to refer to by number.

LEMMA 1.—*If f be a function of x only, which possesses an integral with respect to $g(x, y)$, then*

$$\int_{0,\,0}^{a,\,b} f(x)\, dg(x, y) = \int_0^a f(x) d\Big[g(x, y)\Big]_0^b, \tag{1}$$

where, in accordance with the notation explained, the square bracket means the increment of $g(x, y)$ taken with respect to y from 0 to b.

Here x may be a single variable, or an ensemble of variables x, and y may be a single variable, or an ensemble of variables y.

LEMMA 2.—*If the function with respect to which we are integrating, say $g(x)$, is an integral with respect to x, say*

$$g(x) = \int h(x) dx,$$

then

$$\int f(x)\, dg(x) = \int f(x) h(x) dx. \tag{2}$$

Here x may stand for a single variable, or for an ensemble of variables. In the latter case x, in the symbol dx, stands for the product of the ensemble of variables.

In the third Lemma we use the Riemann expression for an integral which in two dimensions is the following :—

$$\int_{0,\,0}^{a,\,b} f(x, y) dg(x, y) = \mathrm{Lt} \sum_{i=1}^{m-1} \sum_{j=1}^{n-1} f(P_{i,j})(g_{i,j} - g_{i,j+1} - g_{i+1,j} + g_{i+1,j+1}). \tag{3}$$

Here the rectangle of integration (0, 0 ; a, b) is supposed to be divided up into partial rectangles by means of $m-2$ parallels to the axis of x and $n-2$ parallels to the axis of y. Denoting the values of $g(x, y)$ at the angular points of the partial rectangles in double order by

$$g_{i,j} \quad (i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n),$$

and by $P_{i,j}$ any point of the closed partial rectangle whose left-hand bottom corner is the point at which $g(x, y) = g_{i,j}$, the double summation on the right is of the form used in forming a Riemann integral, and reduces to that leading to a Riemann integral when the function $g(x, y)$ is

the product xy. The limit is to be taken in the usual way, when the number of dividing lines is indefinitely increased, and the distances between consecutive parallels is indefinitely diminished.

The necessary and sufficient condition that $f(x, y)$ should possess a "Riemann integral" with respect to a function $g(x, y)$ of bounded variation, is that the variation of $g(x, y)$ over the set of points at which $f(x, y)$ is discontinuous should be zero.

This enables us to state our Lemma in the following form :—

LEMMA 3.—*Any function of bounded variation has a "Riemann integral" with respect to any continuous function of bounded variation.*

It is only necessary to give the proof for two dimensions. In fact the variation of a continuous function with respect to points lying on a single parallel to either axis is clearly zero ; therefore the same is true of points lying on a finite number of such lines. But the discontinuities of a function of bounded variation lie on a countable set of such lines ;* therefore the variation of the continuous function with respect to these discontinuities is not greater than its variation with respect to those lines, which, being the limit of the variation with respect to a finite number of the lines, is zero.†

5. This last result enables us to prove the Formula for Integration by Parts for an integral with respect to a function $g(x, y)$ of bounded variation. The formula is as follows :—

$$\int_{0,0}^{a,b} g(x,y)\,df(x,y) = \Big[f(x,y)\,g(x,y)\Big]_{0,0}^{a,b} - \int_0^a \Big[f\,dg\Big]_{y=0}^{b} - \int_0^b \Big[f\,dg\Big]_{x=0}^{a} + \int_{0,0}^{a,b} f(x,y)\,dg(x,y), \qquad (4)$$

and is certainly valid when f and g are both functions of bounded variation, and one of them is continuous.

To prove this we start with the following algebraic identity, which is

* See a forthcoming communication to the Royal Society.

† More generally any function whose discontinuities lie on a countable set of straight lines has a "Riemann integral" with respect to any continuous function of bounded variation. Such a function is the proper generalisation for the plane of a function of a single variable with a countable set of discontinuities.

easily verified :—

$$\sum_{i=1}^{m-1}\sum_{j=1}^{n-1}[g_{i,j}(f_{i,j}-f_{i+1,j}-f_{i,j+1}+f_{i+1,j+1}) \\ -f_{i+1,j+1}(g_{i,j}-g_{i+1,j}-g_{i,j+1}-g_{i+1,j+1})]$$

$$=\sum_{i=1}^{m-1} f_{i+1,1}(g_{i+1,1}-g_{i,1})+\sum_{j=1}^{n-1} f_{1,j+1}(g_{1,j+1}-g_{1,j})-\sum_{i=1}^{m-1} f_{i+1,n}(g_{i+1,n}-g_{i,n}) \\ -\sum_{j=1}^{n-1} f_{m,j+1}(g_{m,j+1}-g_{m,j})+f_{1,1}g_{1,1}-f_{1,n}g_{1,n}-f_{m,1}g_{m,1}+f_{m,n}g_{m,n}.$$

Now suppose that $f(x, y)$ and $g(x, y)$ are two monotonely monotone functions, such that each has a "Riemann" integral with respect to the other, then dividing the rectangle of integration $(0, 0 ; a, b)$ up into partial rectangles, and applying the formula (3), where for $g(P_{i,j})$ we take the value $f_{i,j}$ at the left-hand bottom corner of the rectangle under consideration, and for $f(P_{i,j})$ we take the value $f_{i+1,j+1}$ at the top right-hand corner, we have on the left-hand side of our algebraic identity the difference of the two expressions whose limit, by (3), gives the corresponding integral. Thus in the limit, when the upper bound of the length of the sides of the partial rectangles approaches zero, our identity becomes

$$\int_{0,0}^{a,b} g(x, y)\, df(x, y)-\int_{0,0}^{a,b} f(x, y)\, dg(x, y)$$

$$=\int_0^a f(x, 0)\, dg(x, 0)+\int_0^b f(0, y)\, dg(0, y)-\int_0^a f(x, b)\, dg(x, b) \\ -\int_0^b f(a, y)\, dg(a, y)+\Big[f(x, y)\, g(x, y)\Big]_{0,0}^{a,b},$$

which is the same as the formula (4) to be proved.

6. Using (1), we have

$$\int_{0,0}^{a,b} f(x, b)\, dg(x, y) = \int_0^a f(x, b)\,[dg(x, b)-dg(x, 0)].$$

Therefore

$$-\int_0^a \Big[f(x, y)\, dg(x, y)\Big]_{y=0}^{b}$$

$$=-\int_{0,0}^{a,b} f(x, b)\, dg(x, y)-\int_0^a [f(x, b)-f(x, 0)]\, dg(x, 0)$$

$$=-\int_{0,0}^{a,b} f(x, b)\, dg(x, y)+\int_0^a \Big[f(x, y)\Big]_{x,0}^{a,b} dg(x, 0)$$

$$-[f(a, b)-f(a, 0)][g(a, 0)-g(0, 0)].$$

A similar expression for

$$-\int_0^b \Big[f(x, y)\, dg(x, y) \Big]_{x=0}^{a}$$

may be at once written down.

Substituting these expressions in (4), we get the following :—

$$\int_{0,\,0}^{a,\,b} g(x, y)\, df(x, y)$$

$$= g(0, 0) \Big[f(x, y) \Big]_{0,\,0}^{a,\,b} + \int_{0,\,0}^{a,\,b} \Big[f(x, y) \Big]_{x,\,y}^{a,\,b} dg(x, y)$$

$$+ \int_0^a \Big[f(x, y) \Big]_{x,\,0}^{a,\,b} dg(x, 0) + \int_0^b \Big[f(x, y) \Big]_{0,\,y}^{a,\,b} dg(0, y). \quad (5)$$

Similarly we have the following variants of the same formula :—

$$\int_{0,\,0}^{a,\,b} g(x, y)\, df(x, y)$$

$$= g(a, b) \Big[f \Big]_{0,\,0}^{a,\,b} + \int_{0,\,0}^{a,\,b} \Big[f \Big]_{0,\,0}^{x,\,y} dg(x, y) - \int_0^a \Big[f \Big]_{0,\,0}^{x,\,b} dg(x, b) - \int_0^b \Big[f \Big]_{0,\,0}^{a,\,y} dg(a, y), \quad (6)$$

$$\int_{0,\,0}^{a,\,b} g(x, y)\, df(x, y)$$

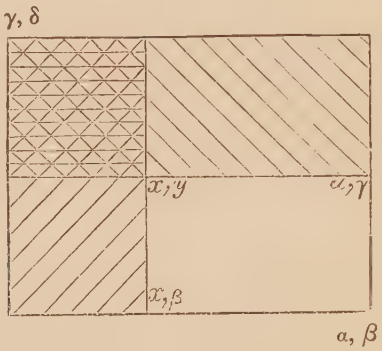
$$= g(0, b) \Big[f \Big]_{0,\,0}^{a,\,b} - \int_{0,\,0}^{a,\,b} \Big[f \Big]_{x,\,0}^{a,\,y} dg(x, y) + \int_0^a \Big[f \Big]_{x,\,0}^{a,\,b} dg(x, b) - \int_0^b \Big[f \Big]_{0,\,0}^{a,\,y} dg(0, y), \quad (7)$$

$$\int_{0,\,0}^{a,\,b} g(x, y)\, df(x, y)$$

$$= g(a, 0) \Big[f \Big]_{0,\,0}^{a,\,b} - \int_{0,\,0}^{a,\,b} \Big[f \Big]_{0,\,y}^{x,\,b} dg(x, y) - \int_0^a \Big[f \Big]_{0,\,0}^{x,\,b} dg(x, b) + \int_0^b \Big[f \Big]_{0,\,y}^{a,\,b} dg(0, y). \quad (8)$$

7. If we examine these formulæ, we shall see that the arguments of the function g on the right are the angular points of a certain one of the rectangles determined in $(0, 0 ; a, b)$ by the parallels through (x, y) to the sides. This rectangle is also that to which the first square bracket on the right refers. Shading the rectangles corresponding to the other three square brackets, the first horizontally, the second by descending, and the

third by ascending lines, we get such a figure as that appended; the first rectangle is thus left plain.



In fact all four formulæ (5)–(8) are summed up in the following, in which (α, β) and (γ, δ) are any two opposite corners of the rectangle $(0, 0; a, b)$:—

$$\int_{\alpha,\beta}^{\gamma,\delta} g\,df = g(\alpha,\beta)\Big[f\Big]_{\alpha,\beta}^{\gamma,\delta} + \int_{\alpha,\beta}^{\gamma,\delta}\Big[f\Big]_{x,y}^{\gamma,\delta} dg(x,y) + \int_{\alpha}^{\gamma}\Big[f\Big]_{x,\beta}^{\gamma,\delta} dg(x,\beta) + \int_{\beta}^{\delta}\Big[f\Big]_{\alpha,y}^{\gamma,\delta} dg(\alpha,y). \quad (9)$$

Symbolically we may write this formula in the following form:—

$$\int_{\alpha}^{\beta} g\,df = \sum_{i=1}^{4}\int_{\alpha}^{\beta}\Big[f\Big]_{\xi_i}^{\beta} dg(\xi_i), \quad (10)$$

where, x_1 and x_2 being the coordinates, ξ denotes the point (x_1, x_2), and α and β are any two opposite points of the fundamental rectangle. The points ξ_i are supposed to be the angular points of the rectangle (α, ξ). One of these is the point ξ itself, and the corresponding term in (10) is the second term of (9). Another of the points ξ_i is the point α, and the corresponding symbolic term in (10) must be understood to mean the product of $g(\alpha)$ into $\Big[f\Big]_{\alpha}^{\beta}$, giving thus the first term of (9). The remaining two points ξ_i are the projections of ξ on the parallels to the axes through the point α, and the corresponding symbolic terms must be understood to mean the integrals along those parallels of $\Big[f\Big]_{\xi_i}^{\beta}$ with respect to $g(\xi_i)$; in other words, the double integral of the function of a single variable,

which appears formally in our expression, is to be interpreted as a single integral, the superfluous coordinates in the ensembles (a_1, a_2), or α, and (b_1, b_2), or β, being simply omitted; these two symbolic expressions, properly interpreted, give us therefore the two remaining terms of (9).

8. The corresponding formulæ for Integration by Parts when there are more variables are now evident. For three variables we have, corresponding to (4), the three dimensional formula for Integration by Parts:—

$$\int_{0,0,0}^{a,b,c} g(x,y,z)\,df(x,y,z)$$

$$= \Big[fg\Big]_{0,0,0}^{a,b,c} - \int_0^a \Big[f\,dg\Big]_{0,0}^{b,c} - \int_0^b \Big[f\,dg\Big]_{0,0}^{a,c} - \int_0^c \Big[f\,dg\Big]_{0,0}^{a,b}$$

$$+ \int_{0,0}^{a,b} \Big[f\,dg\Big]_0^c + \int_{0,0}^{b,c} \Big[f\,dg\Big]_0^a + \int_{0,0}^{a,c} \Big[f\,dg\Big]_0^b - \int_{0,0,0}^{a,b,c} f(x,y,z)\,dg(x,y,z). \quad (11)$$

Here the first term on the right is the three-dimensional increment of fg. The three-dimensional increment of a function $f(x, y, z)$ is a summation of which the first term is $f(a, b, c)$, and the subsequent terms are got from this by putting one or more of a, b, and c equal to zero, and for each such substitution effecting a change of sign. Thus the three terms of the type $f(a, b, 0)$ and the term $f(0, 0, 0)$ will have the sign $-$, and the remaining terms, which are of the type $f(a, 0, 0)$, the sign $+$.

Hence it follows that, when an increment is written in the square bracket notation $\Big[f\Big]_{0,0,0}^{a,b,c}$, we must change the sign if we interchange any one of the letters a, b, or c with one of the zeros; for instance,

$$\Big[f\Big]_{0,0,0}^{a,b,c} = -\Big[f\Big]_{0,0,c}^{a,b,0} = \Big[f\Big]_{0,b,c}^{a,0,0}.$$

A similar change of sign must be made if we make such a change in the limits of the integral; thus

$$\int_{0,0,0}^{a,b,c} f\,dg. = -\int_{0,b,0}^{a,0,c} f\,dg = \int_{0,0,c}^{a,0,0} f\,dg.$$

The formula corresponding to the (α, β)-formula (9) is accordingly the

following :—

$$\int_{\alpha_1,\,\alpha_2,\,\alpha_3}^{\beta_1,\,\beta_2,\,\beta_3} g\,df = g(\alpha_1,\,\alpha_2,\,\alpha_3)\left[f\right]_{\alpha_1,\,\alpha_2,\,\alpha_3}^{\beta_1,\,\beta_2,\,\beta_3} + \int_{\alpha_1,\,\alpha_2,\,\alpha_3}^{\beta_1,\,\beta_2,\,\beta_3}\left[f\right]_{x_1,\,x_2,\,x_3}^{\beta_1,\,\beta_2,\,\beta_3} dg(x_1,\,x_2,\,x_3)$$
$$+\int_{\alpha_1,\,\alpha_2}^{\beta_1,\,\beta_2}\left[f\right]_{x_1,\,x_2,\,\alpha_3}^{\beta_1,\,\beta_2,\,\beta_3} dg(x_1,x_2,\alpha_3)+\ldots+\ldots$$
$$+\int_{\alpha_1}^{\beta_1}\left[f\right]_{x_1,\,\alpha_2,\,\alpha_3}^{\beta_1,\,\beta_2,\,\beta_3} dg(x_1,\,\alpha_2,\,\alpha_3)+\ldots+\ldots, \tag{12}$$

which we may write symbolically, in the form corresponding to (19),

$$\int_\alpha^\beta g\,df = \sum_{i=1}^{8}\int_\alpha^\beta\left[f\right]_{\xi_i}^\beta dg(\xi_i), \tag{13}$$

where, x_1, x_2, and x_3 being the coordinates, ξ denotes the point (x_1, x_2, x_3), and α and β are any two opposite angular points of the fundamental parallelepiped. The points ξ_i denote the angular points of the parallelepiped (α, ξ).

As already explained in relation to the formula (10), the symbolic expression $\int_\alpha^\beta\left[f\right]_{\xi_i}^\beta dg(\xi_i)$ must be interpreted to mean an integral of the number of dimensions dictated by the number of variables in ξ_i, the coordinates in α and β corresponding to the constant coordinates in ξ_i being simply omitted from the limits of integration; in the case when ξ_i contains no variable at all, the symbolic expression is to be interpreted as the product of $g(\xi_i)$ into the integrand.

Thus, for instance, we have the following specimens of the eight formulæ which result from (12) or (13), when we assign to α its different positions.

$$\int_{0,\,0,\,0}^{a,\,b,\,c} fgd(xyz)$$
$$= g(0,\,0,\,0)\left[f\right]_{0,\,0,\,0}^{a,\,b,\,c} + \int_{0,\,0,\,0}^{a,\,b,\,c}\left[f\right]_{x,\,y,\,z}^{a,\,b,\,c} dg(x,\,y,\,z) + \int_{0,\,0}^{b,\,c}\left[f\right]_{0,\,y,\,z}^{a,\,b,\,c} dg(0,\,y,\,z)$$
$$+\int_{0,\,0}^{a,\,c}\left[f\right]_{x,\,0,\,z}^{a,\,b,\,c} dg(x,\,0,\,z)+\int_{0,\,0}^{a,\,b}\left[f\right]_{x,\,y,\,0}^{a,\,b,\,c} dg(x,\,y,\,0)$$
$$+\int_{0}^{a}\left[f\right]_{x,\,0,\,0}^{a,\,b,\,c} dg(x,\,0,\,0)+\int_{0}^{b}\left[f\right]_{0,\,y,\,0}^{a,\,b,\,c} dg(0,\,y,\,0)$$
$$+\int_{0}^{c}\left[f\right]_{0,\,0,\,z}^{a,\,b,\,c} dg(0,\,0,\,z) \tag{14}$$

and

$$\int_{0,0,0}^{a,b,c} f\,gd(xyz)$$

$$= g(a,b,c)\left[f\right]_{0,0,0}^{a,b,c} - \int_{0,0,0}^{a,b,c}\left[f\right]_{0,0,0}^{x,y,z} dg(x,y,z) + \int_{0,0}^{b,c}\left[f\right]_{0,0,0}^{a,y,z} dg(a,y,z)$$

$$+ \int_{0,0}^{c,c}\left[f\right]_{0,0,0}^{a,b,z} dg(x,b,z) + \int_{0,0}^{a,b}\left[f\right]_{0,0,0}^{a,x,y} dg(a,x,y)$$

$$- \int_{0}^{a}\left[f\right]_{0,0,0}^{x,b,c} dg(x,b,c) - \int_{0}^{b}\left[f\right]_{0,0,0}^{a,y,c} dg(a,y,c)$$

$$- \int_{0}^{c}\left[f\right]_{0,0,0}^{a,b,z} dg(a,b,z), \tag{15}$$

and

$$\int_{0,0,0}^{a,b,c} f\,gd(xyz)$$

$$= g(0,b,c)\left[f\right]_{0,0,0}^{a,b,c} + \int_{0,0,0}^{a,b,c}\left[f\right]_{x,0,0}^{a,y,z} dg(x,y,z) + \int_{0,0}^{b,c}\left[f\right]_{0,0,0}^{a,y,z} dg(0,y,z)$$

$$- \int_{0,0}^{a,c}\left[f\right]_{x,0,0}^{a,b,z} dg(x,b,z) - \int_{0,0}^{a,b}\left[f\right]_{x,0,0}^{a,y,c} dg(x,y,c)$$

$$+ \int_{0}^{a}\left[f\right]_{x,0,0}^{a,b,c} dg(x,b,c) - \int_{0}^{b}\left[f\right]_{0,0,0}^{a,y,c} dg(0,y,c)$$

$$- \int_{0}^{c}\left[f\right]_{0,0,0}^{a,b,z} dg(0,b,z). \tag{16}$$

9. The corresponding formulæ for n-dimensions are now easily found. The general formula for Integration by Parts in n-dimensions is as follows; for brevity I use ξ for $(x_1, x_2, \ldots, x_n)$, α for $(a_1, a_2, \ldots, a_n)$, and β for $(b_1, b_2, \ldots, b_n)$:—

$$\int_{\alpha}^{\beta} g\,df = \left[fg\right]_{\alpha}^{\beta} - \sum_{i}\int_{a_i}^{b_i}\left[f\,dg\right]_{a_1, a_2, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n}^{b_1, b_2, \ldots, b_{i-1}, b_{i+1}, \ldots, b_n}$$

$$+ \sum_{i,j}\int_{a_i, a_j}^{b_i, b_j}\left[f\,dg\right]_{a_1, a_2, \ldots, a_{i-1}, a_{i+1}, \ldots, a_{j-1}, a_{j+1}, \ldots, a_n}^{b_1, b_2, \ldots, b_{i-1}, b_{i+1}, \ldots, b_{j-1}, b_{j+1}, \ldots, b_n}$$

$$- \sum_{i,j,k} + \sum_{i,j,k,l} + \ldots$$

$$+ (-)^n \int_{\alpha}^{\beta} f(\xi)\,dg(\xi). \tag{17}$$

This may then be put in forms corresponding to the various variants of the (α, β)-formula, which may be written symbolically

$$\int_\alpha^\beta g\, df = \sum_{i=1}^{2^n} \int_\alpha^\beta \Big[f\Big]_{\xi_i}^\beta dg(\xi_i), \tag{18}$$

the points ξ_i denoting the angular points of the n-dimensional rectangle (α, ξ), and the symbolic integral of a function of only r of the coordinates with respect to a function of the same variables being interpreted as the corresponding r-dimensional integral, the superfluous coordinates in the limits of integration being suppressed; when the integration is that of a constant integrand with respect to a constant, the symbol is to be interpreted as the product of the latter constant into the integrand.

10. If $g(x, y)$ is monotone increasing with respect to (x, y), to x and to y, each of the three integrals on the right of the equation (5) is less than or equal to the corresponding integral in which the integrand is replaced by U, where U is the upper and L the lower bound of $\Big[f(x, y)\Big]_{x, y}^{a, b}$, and it is greater than or equal to the same integral with L for integrand. Hence the sum of the three integrals is less than or equal to $U[g(a, b)-g(0, 0)]$, and greater than or equal to $L[g(a, b)-g(0, 0)]$.

Thus, denoting by M a certain value between L and U inclusive, we get, from (5),

$$\int_{0, 0}^{a, b} g(x, y)\, df(x, y) = g(0, 0)\Big[f(x, y)\Big]_{0, 0}^{a, b} + M[g(a, b)-g(0, 0)]. \tag{19}$$

If $g(0, 0)$ is positive, that is if $g(x, y)$ is a positive function, the first term on the right is also less than or equal to $Ug(0, 0)$, and greater than or equal to $Lg(0, 0)$, so that, adding this term up with the others, we get, similarly, from (5),

$$\int_{0, 0}^{a, b} g(x, y)\, df(x, y) = Mg(a, b). \tag{20}$$

In particular, if f is such that $\Big[f(x, y)\Big]_{x, y}^{a, b}$ assumes all values between its upper and lower bounds, which will in particular be the case if f is a continuous function, and as hitherto, monotonely monotone, we may replace M in these equations by $\Big[f(x, y)\Big]_{X, Y}^{a, b}$, where (X, Y) is some point of the closed rectangle $(0, 0; a, b)$.

The equations thus obtained, viz.,

$$\int_{0,0}^{a,b} g(x,y)\,df(x,y) = g(0,0)\Big[f(x,y)\Big]_{0,0}^{a,b} + [g(a,b)-g(0,0)]\Big[f(x,y)\Big]_{X,Y}^{a,b}, \quad (21)$$

which holds when $g(x,y)$ is monotone ascending with respect to (x,y), x, and y, and

$$\int_{0,0}^{a,b} g(x,y)\,df(x,y) = g(a,b)\Big[f(x,y)\Big]_{X,Y}^{a,b}, \quad (22)$$

in which in addition g is supposed to be a positive function, lead at once to the Second Theorem of the Mean, to be given in the next article, and are themselves Mean Value Theorems of a more general nature, $f(x,y)$ being in the case of the Second Theorem of the Mean an absolutely convergent integral, and in the present article only a function of bounded variation, which is continuous or at least such that $\Big[f(x,y)\Big]_{x,y}^{a,b}$ assumes all values between its upper and lower bounds in $(0,0;\ a,b)$; $g(x,y)$ is a positive function, all of whose increments are positive, and, if f is not continuous, g is continuous, or, more generally, f and g are such that each has zero variation with respect to the discontinuities of the other.

11. Variants of the last two equations can be written down, using (6), (7), and (8) instead of (5), applicable when $g(x,y)$ has its three increments, with respect to (x,y), x, and y respectively, of the same sign as that preceding the first, second, and third integral in the equation used.

It will be seen that, in the case already written out, in which the equation (5) was used, the increments of $g(x,y)$ have to be of the same signs as those of xy, when x and y lie between zero and unity. When (6) is used, the increments of $g(x,y)$ have to be of the same sign as those of $(1-x)(1-y)$, when (7) is used the same as those of $x(1-y)$, and when (8) is used the same as those of $y(1-x)$.

12. The function

$$F(x,y) = \int_{0,0}^{x,y} f(x,y)\,dx\,dy$$

is a continuous function of bounded variation, so that the formula of Integration by Parts certainly holds when we replace f by F, g being any

function of bounded variation. Also $\Big[F(x, y)\Big]_{x, y}^{a, b}$ is the same as

$$\int_{x, y}^{a, b} f(x, y)\, dx\, dy,$$

and therefore assumes all values between its upper and lower bounds L and U. Hence we may put F for f in the equations of the preceding article. But then, as we have seen, the left-hand side takes the form

$$\int_{0, 0}^{a, b} f(x, y)\, g(x, y)\, dx\, dy.$$

Thus we get the following from (20) :—

$$\int_{0, 0}^{a, b} f(x, y)\, g(x, y)\, dx\, dy = g(a, b) \int_{X, Y}^{a, b} f(x, y)\, dx\, dy, \tag{23}$$

provided $g(x, y)$ is a positive monotonely monotone function, all of whose increments are positive. This is the Second Theorem of the Mean for two dimensions.

While, as explained in the preceding article, using in turn the equations (6)–(8) instead of (5), we have the following :—

If $g(x, y)$ is a positive function, monotonely increasing with respect to (x, y), and decreasing with respect to x and with respect to y,

$$\int_{0, 0}^{a, b} f(x, y)\, g(x, y)\, dx\, dy = g(0, 0) \int_{0, 0}^{X, Y} f(x, y)\, dx\, dy. \tag{24}$$

If $g(x, y)$ is a positive function, monotonely decreasing with respect to (x, y) and with respect to x, and increasing with respect to y,

$$\int_{0, 0}^{a, b} f(x, y)\, g(x, y)\, dx\, dy = g(0, b) \int_{0, Y}^{X, b} f(x, y)\, dx\, dy. \tag{25}$$

If $g(x, y)$ is a positive function, monotonely decreasing with respect to (x, y) and with respect to y, and increasing with respect to x,

$$\int_{0, 0}^{a, b} f(x, y)\, g(x, y)\, dx\, dy = g(a, 0) \int_{X, 0}^{a, Y} f(x, y)\, dx\, dy. \tag{26}$$

These four equations exactly take the place in two dimensions of the two given by Bonnet in one dimension.

12. In three dimensions we have in like manner the Second Theorem of the Mean as follows :—

If $g(x, y, z)$ is a positive function all of whose increments, three, two, and one dimensional, are positive, we have

$$\int_{0,0,0}^{a,b,c} f(x,y,z)\, g(x,y,z)\, d(xyz) = g(a,b,c) \int_{X,Y,Z}^{a,b,c} f(x,y,z)\, d(xyz). \quad (27)$$

This has of course seven equivalent variants, of which the following are typical :—

If all the increments of $g(x, y, z)$ are negative, except the two-dimensional ones which are positive, then

$$\int_{0,0,0}^{a,b,c} fg\, d(x,y,z) = g(0,0,0) \int_{0,0,0}^{X,Y,Z} f(x,y,z)\, d(xyz). \quad (28)$$

If all the increments of $g(x, y, z)$ are negative, except the (x, y, z), (y, z) and x increments, which are all positive, then

$$\int_{0,0,0}^{a,b,c} fg\, d(xyz) = g(a,0,0) \int_{x,0,0}^{a,Y,Z} f(x,y,z)\, d(xyz). \quad (29)$$

If all the increments of $g(x, y, z)$ are negative, except the (y, z), y and z increments, which are positive, then

$$\int_{0,0,0}^{a,b,c} fg\, d(xyz) = g(0,b,c) \int_{0,Y,Z}^{X,b,c} f(x,y,z)\, d(xyz). \quad (30)$$

Similar equations hold for any number of variables.

13. Without any assumption as to the sign of $g(x, y)$, the equation (19) gives us the following :—

$$\int_{0,0}^{a,b} f(x,y)\, g(x,y)\, dx\, dy)$$
$$= g(0,0) \int_{0,0}^{a,b} f(x,y)\, dx\, dy + [g(a,b) - g(0,0)] \int_{X,Y}^{a,b} f(x,y)\, dx\, dy, \quad (27)$$

where (X, Y) is some point of the rectangle $(0, 0\,;\ a, b)$; here $g(x, y)$ has all its increments positive, and is otherwise perfectly general, and $f(x, y)$ is merely a summable function.

This is another form of the Second Theorem of the Mean, and has again three variants, which are easily written down. Each of these equations also holds when $-g(x, y)$ has the properties attributed to $g(x, y)$.

The corresponding equations when there are more than two variables are of precisely the same form.

14. The forms of the Theorem of the Mean which we have hitherto obtained are such that, whatever monotone function $g(x, y)$ may be, we have an appropriate formula. On the other hand, if we substitute $F(x, y)$ for $f(x, y)$ directly in the formula (4) for Integration by Parts, we get a single expression for the integral which appears on the left in the equations of the Theorem of the Mean, which is valid whatever function of bounded variation $g(x, y)$ may be. Since $F(x, y)$ is zero whenever x or y is zero, this expression is as follows :—

$$\int_{0,0}^{a,b} f(x, y)\, g(x, y)\, dx\, dy$$

$$= F(a, b)\, g(a, b) - \int_0^a F(x, b)\, dg(x, b) - \int_0^b F(a, y)\, dg(a, y)$$

$$+ \int_{0,0}^{a,b} F(x, y)\, dg(x, y). \tag{28}$$

This equation may often enable us to dispense with the Theorem of the Mean. If, however, we desire a form of the Theorem of the Mean, available for any type of function of bounded variation, we may proceed as follows.

If $g(x, y)$ is a function of bounded variation, we have*

$$g(x, y) = g(0, 0) = \pi(x, y) - \nu(x, y),$$

where $\pi(x, y)$ and $\nu(x, y)$ are functions which, as well as all their increments, are positive, and which are both zero at the origin ; in fact $\pi(x, y)$ is the sum of the positive variations of $g(x, y)$ with respect to (x, y), to x and to y, and $\nu(x, y)$ is the sum of the corresponding negative variations.

Denoting therefore by M the upper bound of

$$\left| \int_{x, y}^{a, b} f(x, y)\, dx\, dy \right|,$$

so that M is the greater of the absolute values of the quantities U and L

* G. H. Hardy, *loc. cit.*, p. 59 ; H. Lebesgue, "Sur l'intégration des fonctions discontinues," *Ann. de l'École Normale Supérieure*, Ser. 3, Vol. 27, 1910. See also my paper on "Multiple Fourier Series," *Proc. London Math. Soc.*, Ser. 2, Vol. 11, 1912, pp. 146, 147.

previously employed, we have, by § 10, f being now replaced by F,

$$-M\pi(a, b) \leqslant \int_{0,0}^{a,b} f\pi\, dx\, dy \leqslant M\pi(a, b),$$

and

$$-M(a, b) \leqslant \int_{0,0}^{a,b} f\nu\, dx\, dy \leqslant M\nu(a, b).$$

Hence

$$\left| \int_{0,0}^{a,b} f(x, y)\, g(x, y)\, dx\, dy - g(0, 0) \int_{0,0}^{a,b} f(x, y)\, dx\, dy \right| \leqslant MV(a, b), \quad (29)$$

where

$$V(a, b) = \pi(a, b) + \nu(a, b)$$

is the total variation of $g(x, y)$.

A PROBLEM IN DIOPHANTINE APPROXIMATION

By H. T. J. Norton.

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The purpose of this note is to discuss a problem of which a particular case is this :—if ϑ is a real number, to find an upper limit to the smallest natural number n which is such that the difference between $n^2\vartheta$ and the nearest integer is less than some assigned positive number $1/\lambda$. Mr. Hardy and Mr. Littlewood* have shown that for all values of ϑ and λ an n with the property supposed exists, and that it is less than a number which is independent of ϑ, but they say† that they have not "succeeded in finding a definite function $\Phi(\lambda)$, the same for all ϑ's, such that

$$|\overline{n^2\vartheta}| \leqslant 1/\lambda \quad \text{for} \quad n \leqslant \Phi."$$

The result which is obtained here is that

$$n < \lambda^{2\lambda(1+\epsilon)},$$

where $\epsilon \to 0$ as $\lambda \to \infty$.

The most general theorems of this kind which are known have been proved by Minkowski.‡ One of them is a generalisation of the following proposition to manifolds of many dimensions. Suppose that rectilinear coordinates (X, Y) are taken in a plane and that H is a closed contour in the plane, which contains the origin in its interior, is symmetrical about the origin—that is to say, if (X, Y) lies on H so does $(-X, -Y)$—and is nowhere concave—that is, if P_1 and P_2 are points on H, then every point between P_1 and P_2 on the line joining them lies upon H or in its interior : then if the area contained by H is not less than 4,

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* *Acta Mathematica*, Vol. 37, pp. 155–190. Extensions of the theorems proved by Mr. Hardy and Mr. Littlewood have been published lately by Mr. Fowler in this journal and Herr H. Weyl in the *Annalen ;* they are, however, in a different direction and are not related to the subject of the present note.

† *l.c.*, p. 174. The notation $|\overline{n^2\vartheta}|$ means the difference, taken positively, between $n^2\vartheta$ and the nearest integer.

‡ *Geometrie der Zahlen* (1910), pp. 76, 77, 218, 219.

there are two points with integral coordinates, besides the origin, which lie within or upon the contour. For instance, if H is the parallelogram bounded by the lines

$$X = \pm q, \qquad X\vartheta - Y = \pm \frac{1}{q},$$

the area contained by H is 4, the other conditions are fulfilled and there are, therefore, two integral points besides the origin upon or within the contour. If (ξ, η) is one of them

$$|\xi| \leqslant q, \qquad |\xi\vartheta - \eta| \leqslant \frac{1}{q},$$

and hence follows the well-known theorem:—if ϑ and q are any real numbers, there exists a positive integer n which is less than or equal to q, and is such that the difference between $n\vartheta$ and the nearest integer is less than or equal to $1/q$. Minkowski's second theorem has the same character, but is more complicated.

From these two theorems most of the results known can be inferred very simply—as, for instance, Dirichlet's* theorem that if $\vartheta_1 \ldots \vartheta_m$ are real numbers we can choose an n less than λ^m so that the difference between $n\vartheta_i$ and the nearest integer is less than $1/\lambda$, or Tchebycheff's theorem that, if ϑ is irrational, we can choose n so that the fractional part of $n\vartheta$ differs from any assigned number between 0 and 1 by as little as we like and that for an infinity of values of n the error is less than $3/n$, or Kronecker's generalisation of this, that if $\vartheta_1 \ldots \vartheta_m$ are linearly independent irrationals we can choose n so that the fractional part of each of the numbers $n\vartheta_i$ differs by as little as we like from any arbitrarily assigned numbers between 0 and 1. The theorem of Mr. Hardy and Mr. Littlewood which is indicated above does not, however, appear to follow so easily. In this case, one is led to consider the closed contour H' which is bounded by the two lines

$$X = \pm \Phi$$

and the two parabolas $$X^2\vartheta - Y = \pm \frac{1}{\lambda}.$$

The theorem is that if Φ exceeds a limit which depends only upon λ, there are two integral points, other than the origin, in the interior of H'. Since this contour is concave, it is not possible to apply Minkowski's theorem directly; and on the other hand it does not seem to be easy to alter the condition of non-concavity so as to bring contours like H' into the reach

* For elementary proofs of these theorems see Hardy and Littlewood, *l.c.*, pp. 158–165.

of the theorem, without destroying the whole argument. The proof which is given here is a modification of the argument used by Dirichlet. It leads to an upper limit which increases with great rapidity as λ tends to infinity, and I am very doubtful whether one increasing much more slowly could not be found. But a more exact determination of the limit seems to present serious difficulties.

For the sake of convenience and compactness of expression, I shall use a special notation. Let "$A \equiv B$" mean "A differs from B by an integer", and let η be a symbol which may denote different numbers on different occasions but only those which are not greater than 1. "$A \equiv \eta B$", then, asserts :—there exists an integer X such that

$$-|B| \leqslant A - X \leqslant |B|.$$

The general theorem* of Mr. Hardy and Mr. Littlewood, which is referred to above, may then be stated as follows :—

"If m, κ, and λ are three whole numbers, there exists a function $\Phi(m, \kappa, \lambda)$ such that, if $f_1(X) \ldots f_m(X)$ are any m polynomials in X with real coefficients and of degree not greater than κ, then the congruences

$$n \cdot f_i(n) \equiv \eta/\lambda \quad (i = 1, 2, \ldots, m)$$

are satisfied by some positive integer $n \leqslant \Phi(m, \kappa, \lambda)$."

The particular case of this theorem, in which $\kappa = 0$, is the theorem of Dirichlet, which is mentioned above, and it is known† that we may put

$$\Phi(m, 0, \lambda) = \lambda^m - 1.$$

My object is to find an explicit formula for Φ which is valid for all values of κ. I take the case $\kappa = 1$; the generalisation by induction to an arbitrary value of κ does not present any difficulty.

Let λ be any positive integer and

$$a_i X + b_i \quad (i = 1, 2, \ldots, m)$$

any m linear functions of X with real coefficients. If

$$g_i(X) = X(a_i X + b_i),$$

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* Cf. *l.c.*, Theorems 1 . 21 and 1 . 31.

† I do not know whether it has been determined exactly how small $\Phi(m, 0, \lambda)$ may be supposed to be. Mr. Hardy and Mr. Littlewood (*l.c.*, p. 159) give the value λ^m. The argument used by Minkowski (*l.c.*, p. 108) would give $\lambda^m - 1$. If Dirichlet's argument is used in the slightly modified form in which it is stated below, the value which is found is $\lambda^m - 2^m + 1$. But differences of this magnitude are too small to alter the formulæ which I obtain for higher values of κ.

the problem is to find an upper limit to the smallest positive n which satisfies the congruences

$$g_i(n) \equiv \eta/\lambda \quad (i = 1, 2, \ldots, m). \tag{1}$$

Suppose that $a_1 \ldots a_j \ldots$ is any sequence of integers, and that $p_0 \ldots p_j \ldots$ is a sequence defined as follows:—$p_0 = 1$; p_1 is the smallest positive integer which satisfies

$$p_1 . 2a_i \equiv \eta a_1^{-1} \quad (i = 1, 2, \ldots, m);$$

and generally p_j is the smallest positive integer such that

$$\left.\begin{aligned} p_j . 2a_i &\equiv \eta . a_j^{-1} \\ p_j . 2(1+p_1)a_i &\equiv \eta . a_j^{-1} \\ \cdots \quad \cdots \quad \cdots \quad &\cdots \quad \cdots \\ p_j . 2(1+p_1+\ldots p_{j-1})a_i &\equiv \eta . a_j^{-1} \end{aligned}\right\} \quad (i = 1, 2, \ldots, m).$$

By the theorem of Dirichlet which has been quoted above, p_j exists and

$$a_j^{mj} > p_j > 0. \tag{2}$$

Write, now, $q_s = \sum_0^s p_j$, and it then follows from the definitions that

$$2p_t . q_s . a_i \equiv \eta a_t^{-1} \tag{3}$$

$$(i = 1, 2, \ldots, m;\ s = 0, 1, \ldots;\ t = s+1, s+2, \ldots),$$

and from (2) that

$$0 < q_{t-1} < q_t < 1 + \sum_1^t a_j^{mj}; \tag{4}$$

and, further, in virtue of (3) and the identical equation

$$g_i(X) = g_i(X') + g_i(X-X') + 2(X-X')X'a_i,$$

$$\begin{aligned} g_i(q_t) &= g_i(q_s) + g_i(q_t-q_s) + 2(q_t-q_s)q_s a_i \\ &= g_i(q_s) + g_i(q_t-q_s) + \sum_{r=s+1}^{t} 2p_r q_s a_i \quad (t > s) \\ &\equiv g_i(q_s) + g_i(q_t-q_s) + \eta \sum_{s+1}^{t} a_j^{-1}, \end{aligned}$$

that is to say

$$g_i(q_t) - g_i(q_s) \equiv g_i(q_t-q_s) + \eta \sum_{s+1}^{t} a_j^{-1} \tag{5}$$

$$(i = 1, 2, \ldots, m;\ s = 0, 1, \ldots;\ t = s+1, s+2, \ldots).$$

The result can now be obtained from (4) and (5) by following the line of argument used by Dirichlet.

Let R be an m-dimensional manifold in which the coordinates range from 0 to 1, and, taking any integer μ, suppose the range of each coordinate is divided into μ equal parts; R will then be sub-divided into μ^m parts in such a way that the corresponding coordinates of two points, which lie in the same part, do not differ by more than $1/\mu$. Now R contains 2^m integral points, the coordinates of which are permutations of 0 and 1, and each of them lies on the boundary of one and of only one of the sub-divisions of R; these sub-divisions, therefore, can be separated into two classes, firstly the 2^m "corner" ones, which contain each an integral point, and secondly the remainder which number $\mu^m - 2^m$. Consider those points of R of which the coordinates are congruent to

$$\{g_1(q_t), g_2(q_t), \ldots, g_m(q_t)\} \quad (t = 0, 1, \ldots, \mu^m - 2^m).$$

To each value of t corresponds a point of R and to each, in general, only one point, but there are more than one if any of the numbers $g_i(q_t)$ is an integer; in the latter case, it may be supposed that some particular one is selected. Since there are $\mu^m - 2^m + 1$ values of t, one of two things is true: either there is a value of t, for which the corresponding point of R lies in a "corner" part, and in this case, since each such part contains an integral point,

$$g_i(q_t) \equiv \eta/\mu \quad (i = 1, 2, \ldots, m;\ t \leqslant \mu^m - 2^m);$$

or else there are two distinct values of t, t_1 and t_2, such that the corresponding points both lie in the same sub-division of R, and in this case

$$g_i(q_{t_1}) - g_i(q_{t_2}) \equiv \eta/\mu \quad (i = 1, 2, \ldots, m),$$

and hence by (5)

$$g_i(q_{t_1} - q_{t_2}) \equiv \eta\left(\mu^{-1} + \sum_{t_2+1}^{t_1} a_j^{-1}\right) \quad (0 \leqslant t_2 < t_1 \leqslant \mu^m - 2^m).$$

In either case, it follows from (4) that

$$g_i(n) \equiv \eta\left(\mu^{-1} + \sum_{1}^{\mu^m - 2^m} a_j^{-1}\right) \quad (i = 1, 2, \ldots, m), \tag{6}$$

for some integer n which satisfies the inequalities

$$0 < n < 1 + \sum_{1}^{\mu^m - 2^m} a_j^{mj}. \tag{7}$$

Arbitrary positive values can be assigned to μ and $a_1 \ldots a_j \ldots$ in (6) and (7),

and if values are assigned in any way that makes

$$\mu^{-1}+\sum_{1}^{\mu^m-\nu^m} a_j^{-1} \leqslant \lambda^{-1}, \tag{8}$$

(7) gives an upper limit to the smallest positive n which satisfies (1).

If, for instance, a_j is taken equal to $\lambda^{(n+1)\beta}$ and β is determined so that $\beta \to 1$ and $\lambda^{\beta-1} \to \infty$ as λ tends to infinity, then (8) is satisfied by taking μ equal to $\lambda(1+\epsilon)$, and (7) gives

$$n < \lambda^{m(n+1)\lambda^m(1+\epsilon)}, \tag{9}$$

where $\epsilon \to 0$ as $\lambda \to \infty$. More refined methods of dealing with (7) and (8) appear to lead to results of the same order of magnitude. It follows from this argument that if in the theorem of Mr. Hardy and Mr. Littlewood referred to above κ is equal to 1, then the expression on the right hand side of (9) can be substituted for Φ. The significance of the result is made more evident if Φ is supposed to be given instead of λ. In this form, the theorem asserts that if q be an integer, then among the numbers $1, 2, \ldots, q$ there is at least one, n, such that

$$n . f_i(n) \equiv \eta/(\log q)^{\frac{1}{m}-\delta} \quad (i = 1, 2, \ldots, m),$$

where δ tends to zero as $q \to \infty$. It is easy to show that the order of δ is at any rate not greater than $\log\log\log q/m \log\log q$.

Corresponding results for an arbitrary value of κ can be obtained by induction. Assuming that Φ is known for $\kappa = \kappa_0 - 1$, the argument for the case $\kappa = \kappa_0$ proceeds step by step as in the case $\kappa = 1$; the only difference consists in the relations which define p_j and the identity from which (5) is deduced.

The identity which is required in the general case is merely that if $g(X)$ is a polynomial in X from which the absolute term is missing and if

$$y_t = \sum_{0}^{t} X_j,$$

then $g(y_t) - g(y_s) - g(y_t - y_s)$

$$= \sum_{s+1}^{t} X_j \left[\{ g(y_j) - g(y_{j-1}) - g(y_j - y_s) + g(y_{j-1} - y_s) \} / X_j \right] \quad (t > s).$$

If g is of degree κ_0, the function between square brackets on the right hand side is a polynomial of degree $\kappa_0 - 2$ in X_j, the coefficients of which are polynomials in y_{j-1} and y_s. If it is written equal to $h_i(X_j, y_{j-1}, y_s)$,

then the congruences which define p_j are

$$p_j h_i(p_j, q_{j-1}, q_s) \equiv \eta \alpha_j^{-1} \quad (i = 1, 2, \ldots, m;\ s = 0, 1, \ldots, j-1),$$

and we have instead of (2)

$$\Phi(mj, \kappa_0 - 1, \alpha_j) > p_j > 0,$$

and instead of (7) $$0 < n < \sum_{1}^{\mu''' - 2^m} \Phi(mj, \kappa_0 - 1, \alpha_j) + 1.$$

Writing $$l_0 q = q, \quad l_1 q = \log q, \quad l_2 q = \log\log q, \quad \ldots,$$

the general theorem is as follows:—

"If m, q, and κ are integers and if $f_1(X) \ldots f_m(X)$ are m arbitrary polynomials in X with real coefficients and of degree not greater than κ, then among the numbers 1, 2, ..., q there is at least one, n, which satisfies the congruences

$$n \,.\, f_i(n) \equiv \eta / (l_\kappa q)^{\frac{1}{m} - \delta} \quad (i = 1, 2, \ldots, m), \tag{10}$$

where δ tends to zero as q tends to infinity and is of order not greater than $l_{\kappa+2} q / m l_{\kappa+1} q$."

The value of $\Phi(m, \kappa, \lambda)$ which is implied by (10) increases so rapidly, as $\lambda \to \infty$, that it appears improbable that this result could not be substantially improved, but I have not been able to prove that it could. In the case $m = 1$, $\kappa = 0$, we can take $\Phi(1, 0, \lambda) = \lambda - 1$, and we cannot take $\Phi(1, 0, \lambda)$ less than $\lambda - 1$, since if ϑ is equal to $1/\lambda - 1$ then $r\vartheta$ is not congruent to η/λ for any r which is less than $\lambda - 1$. In the case $m = 1$, $\kappa = 1$, it can be shewn that we can *not* take $\Phi(1, 1, \lambda) = C\lambda$, where C is constant; that is to say, that for all values of C there are an infinity of λ's and ϑ's for which the difference between $r^2\vartheta$ and the nearest integer is greater than $1/\lambda$ if r is less than or equal to $C\lambda$. More than this I have not been able to prove.

A SYMMETRICAL RELATION BETWEEN LEGENDRE'S FUNCTIONS WITH PARAMETERS COSH α AND COTH α

By F. J. W. WHIPPLE.

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1. The relation

$$e^{-m\iota\pi} Q_n^m(\cosh\alpha) = \sqrt{\left(\frac{\pi}{2}\right)} \frac{\Pi(m+n)}{\sqrt{\sinh\alpha}} P_{-m-\frac{1}{2}}^{-n-\frac{1}{2}}(\coth\alpha), \tag{1}$$

which is discussed in the following notes, was arrived at in the examination of certain integrals which occur in the determination of Green's function for a wedge.* It is found to explain the symmetry of various parts of the theory of Legendre's functions. Its geometrical interpretation is found in the applications of inversion to potential problems connected with toroidal coordinates.

2. It is convenient to write $\cosh\alpha = \mu$, $\coth\alpha = \nu$, so that μ and ν satisfy

$$(\mu^2-1)(\nu^2-1) = 1. \tag{2}$$

We shall show in the first place that if $V_n^m(\mu)$ is any solution of Legendre's equation with μ as variable, then there is a corresponding function $W_{-m-\frac{1}{2}}^{-n-\frac{1}{2}}(\nu)$, which is a solution of the equation with ν as variable, and is connected with $V_n^m(\mu)$ by the equation

$$V_n^m(\mu) = (\nu^2-1)^{\frac{1}{4}} W_{-m-\frac{1}{2}}^{-n-\frac{1}{2}}(\nu).$$

Legendre's differential equation is

$$\left[-\frac{\partial}{\partial\mu}(\mu^2-1)\frac{\partial}{\partial\mu} + n(n+1) + \frac{m^2}{\mu^2-1}\right] V = 0. \tag{3}$$

* *Vide* "Diffraction by a Wedge and Kindred Problems," p. 94 of the present volume, and particularly pp. 107–109.

This may be transformed to

$$\left[-(\nu^2-1)^{\frac{3}{2}}\frac{\partial}{\partial\nu}(\nu^2-1)^{\frac{1}{2}}\frac{\partial}{\partial\nu}+m^2(\nu^2-1)+n(n+1)\right]V=0. \qquad (4)$$

If $(\nu^2-1)^{\frac{1}{4}}W$ is substituted for V, the last equation becomes, after some reduction,

$$\left[-\frac{\partial}{\partial\nu}(\nu^2-1)\frac{\partial}{\partial\nu}+(m-\tfrac{1}{2})(m+\tfrac{1}{2})+\frac{(n+\frac{1}{2})^2}{\nu^2-1}\right]W=0. \qquad (5)$$

Accordingly W is a Legendre function of ν of order $m-\frac{1}{2}$ and rank $n+\frac{1}{2}$. Thus

$$V_n^m(\cosh\alpha)=\sqrt{(\operatorname{cosech}\alpha)}\,W_{m-\frac{1}{2}}^{n+\frac{1}{2}}(\coth\alpha). \qquad (6)$$

Obviously the signs of m and of $n+\frac{1}{2}$ may be changed without altering either of the equations (3) and (5), so that instead of (6) we may write

$$V_n^m(\cosh\alpha)=\sqrt{(\operatorname{cosech}\alpha)}\,W_{-m-\frac{1}{2}}^{-n-\frac{1}{2}}(\coth\alpha).$$

3. The identity (6) could be derived from Olbricht's result that the differential equation (3) is satisfied by either of Riemann's P-functions

$$P\left\{\begin{matrix}0 & \infty & 1 & \\ \frac{1}{2}m & -n & \frac{1}{2}m & \frac{1-\mu}{2} \\ -\frac{1}{2}m & n+1 & -\frac{1}{2}m & \end{matrix}\right\}$$

and

$$P\left\{\begin{matrix}0 & \infty & 1 & \\ -\frac{1}{2}n & m & -\frac{1}{2}n & \frac{\mu+(\mu^2-1)^{\frac{1}{2}}}{2(\mu^2-1)^{\frac{1}{2}}} \\ \frac{1}{2}(n+1) & -m & \frac{1}{2}(n+1) & \end{matrix}\right\}.$$

4. The Legendre functions are defined by different authors in ways which are not consistent. We shall accept the definitions given by Prof. Hobson, to whose paper "On a Type of Spherical Harmonics of Unrestricted Degree, Order and Argument," *Phil. Trans.*, 1896, frequent reference will be made. These definitions are in terms of contour integrals round looped paths. The equation (1) will be derived from a comparison of two of these contour integrals.

The definition of $P_n^m(\cosh\alpha)$ is [*l.c.*, equation (50)]

$$P_n^m(\cosh\alpha)=\frac{1}{2\pi\iota}2^m\frac{\Pi(m-\frac{1}{2})}{\Pi(-\frac{1}{2})}\sinh^m\alpha\int^{e^{\alpha+},\,e^{-\alpha-}}\frac{h^{n+m}}{[(e^\alpha-h)(e^{-\alpha}-h)]^{m+\frac{1}{2}}}dh. \qquad (7)$$

The integral is taken along a looped path which passes round e^{α} anti-clockwise and $e^{-\alpha}$ clockwise. The phases of the factors of the integrand are governed by the following conventions. The phase of h vanishes where the path cuts the real axis on the right of the origin. The phase of $e^{\alpha}-h$ is that of e^{α} at the point where the path cuts the join of the origin to e^{α}, and the phase of $e^{-\alpha}-h$ is determined in a similar way.

The definition of $Q_{n'}^{m'}$ [*l.c.*, equation (9), p. 455] may be written

$$e^{-m'\iota\pi}\,Q_{n'}^{m'}(\coth\alpha)$$

$$= \frac{1}{4\iota\sin n'\pi}\,\frac{\Pi(n'+m')}{\Pi(n')}\,\frac{1}{\sinh^{m'}\alpha}\int^{-1+,\,1-}\frac{1}{2^{n'}}\,\frac{(t^2-1)^{n'}}{(\coth\alpha-t)^{n'+m'+1}}\,dt. \quad (8)$$

Here the phases of $t-1$ and $t+1$ vanish when t is real and greater than 1, and the phase of $\coth\alpha-t$ is such that at the point E, where the line joining $\coth\alpha$ and 1 cuts the path of integration, it is the angle (between $\pm\pi$) the line makes with the positive direction of the real axis of t.

Let us write $h=\cosh\alpha-t\sinh\alpha$ in (7), so that

$$e^{-\alpha}-h=(t-1)\sinh\alpha, \qquad e^{\alpha}-h=(t+1)\sinh\alpha.$$

It follows that

$$P_n^m(\cosh\alpha) = -\frac{1}{2\pi\iota}\,2^m\,\frac{\Pi(m-\frac{1}{2})}{\Pi(-\frac{1}{2})}\sinh^n\alpha\int^{-1+,\,1-}\frac{(\coth\alpha-t)^{n+m}}{(t^2-1)^{m+\frac{1}{2}}}\,dt. \quad (9)$$

In (8) write $\qquad m'=-n-\frac{1}{2}, \qquad n'=-m-\frac{1}{2},$

$$e^{(n+\frac{1}{2})\iota\pi}Q_{-m-\frac{1}{2}}^{-n-\frac{1}{2}}(\coth\alpha) = -\frac{1}{4\iota\sin(m+\frac{1}{2})\pi}\,\frac{\Pi(-n-m-1)}{\Pi(-m-\frac{1}{2})}\sinh^{n+\frac{1}{2}}\alpha$$

$$\times\int^{-1+,\,1-}2^{m+\frac{1}{2}}\,\frac{(\coth\alpha-t)^{n+m}}{(t^2-1)^{m+\frac{1}{2}}}\,dt. \quad (10)$$

Now $\qquad \Pi(m-\frac{1}{2})\,\Pi(-m-\frac{1}{2})\sin(m+\frac{1}{2})\pi=\pi,$

and $\qquad \Pi(-\frac{1}{2}) \qquad\qquad =\sqrt{\pi}.$

Hence, from (9) and (10),

$$e^{(n+\frac{1}{2})\iota\pi}Q_{-m-\frac{1}{2}}^{-n-\frac{1}{2}}(\coth\alpha)=\sqrt{\left(\frac{\pi}{2}\right)}\,\Pi(-m-n-1)\sqrt{(\sinh\alpha)}\,P_n^m(\cosh\alpha), \quad (11)$$

provided that the rules determining the phases in the integrals in equations (9) and (10) are the same.

This can be shown to be the case as long as the real part of $\cosh\alpha$ is

positive, so that the line joining e^{α} and $e^{-\alpha}$ passes to the right of the origin (Fig. 1).

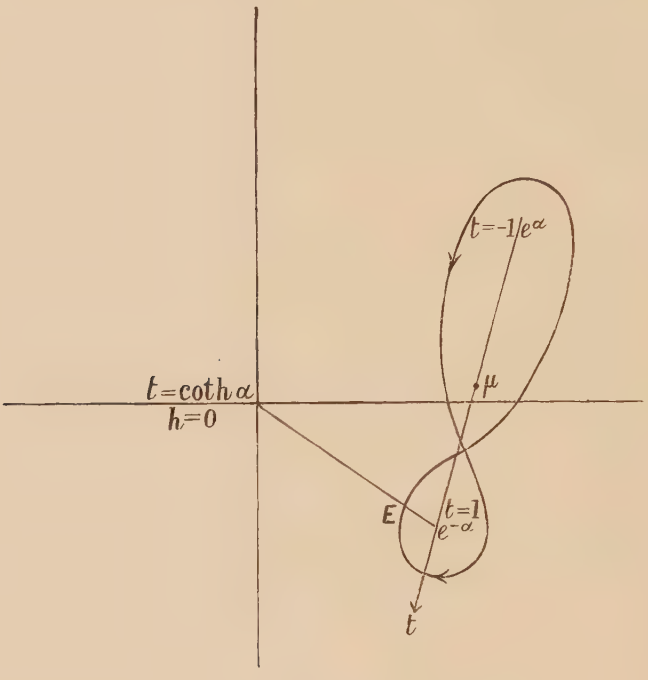

FIG. 1.

If, however, the real part of μ is negative, the definition of $P_n^m(\cosh\alpha)$ implies loops as in Fig. 2*b* passing to the right of the origin, whilst the definition of $Q_{n'}^{m'}(\coth\alpha)$ implies loops, which do not surround the origin, as in Fig. 2*a*. As the two paths can not be reconciled, the equation (11) can not be proved in this case. Accordingly it must be restricted to values of α such that the real parts of $\cosh\alpha$ and $\coth\alpha$ are positive. With this proviso we can write (11) in the form

$$P_n^m(\cosh\alpha) = \sqrt{\left(\frac{2}{\pi}\right)} \frac{1}{\Pi(-m-n-1)} \frac{e^{(n+\frac{1}{2})\iota\pi}}{\sqrt{\sinh\alpha}} Q_{-m-\frac{1}{2}}^{-n-\frac{1}{2}}(\coth\alpha). \qquad (12)$$

By substituting $\cosh\alpha$ for $\coth\alpha$, m for $-n-\frac{1}{2}$, and n for $-m-\frac{1}{2}$, we get

$$e^{-m\iota\pi} Q_n^m(\cosh\alpha) = \sqrt{\left(\frac{\pi}{2}\right)} \Pi(m+n) \frac{1}{\sqrt{\sinh\alpha}} P_{-m-\frac{1}{2}}^{-n-\frac{1}{2}}(\coth\alpha), \qquad (13)$$

which is the relation (1).

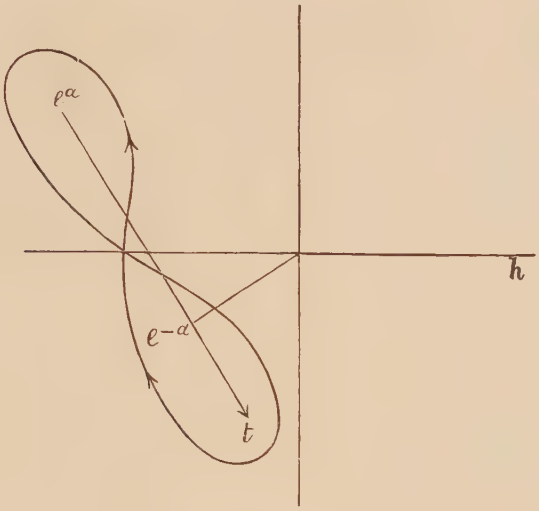

FIG. 2a.

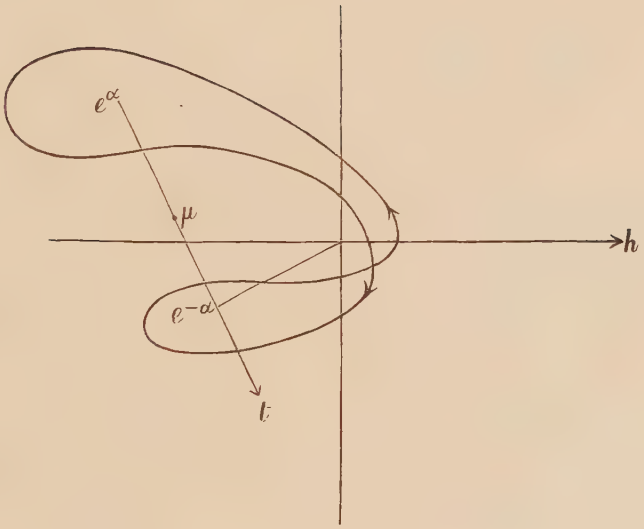

FIG. 2b.

5. The definition adopted for Q in equation (8) suffers from the drawback that it makes Q complex for real values of α. It would be advantageous to take a new definition of Q, so that

$$Q_n^m(\cosh \alpha) = \sqrt{\left(\frac{2}{\pi}\right)} \frac{\Pi(m+n)}{\sqrt{\sinh \alpha}} P^{-n-\frac{1}{2}}_{-m-\frac{1}{2}}(\coth \alpha)$$

could be substituted for (1), when the real parts of $\cosh a$ and $\coth a$ were positive. Hobson's notation has been retained however in what follows.

6. The limitation of the scope of the relations (12) and (13) may be explained by a reference to the domains of $\mu(\equiv \cosh a)$ and of $\nu(\equiv \coth a)$ assumed in the definitions (7) and (8). In each case (Fig. 3) there is a

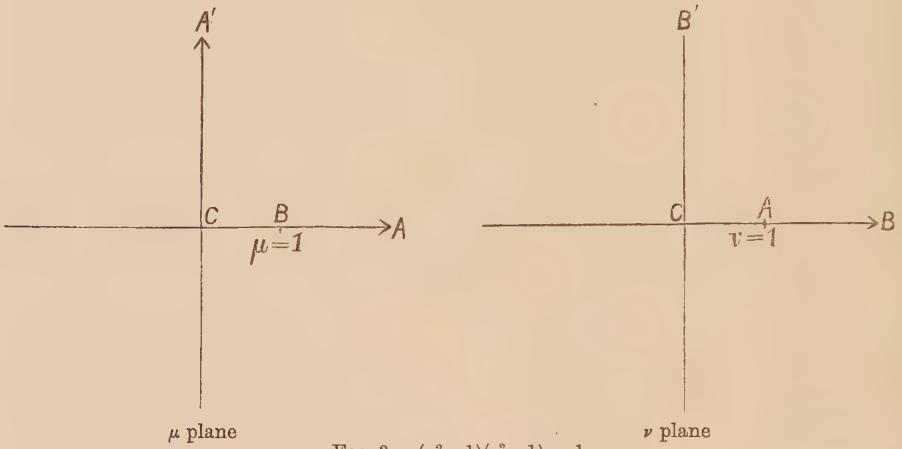

FIG. 3. $(\mu^2-1)(\nu^2-1)=1$.

cross-cut from 1 to $-\infty$ along the real axis. Now the imaginary axis in the domain of μ corresponds to that portion of the real axis in the domain of ν which lies between 0 and 1. Accordingly the analytic functions which are identical in the first quadrant are "continued" across boundaries which do not correspond with each other, and there is no reason to expect identity in the second quadrant.

7. *Formulæ connecting P's and Q's of Associated Orders and Ranks.*

Legendre's equation is satisfied by $P_n^m(\mu)$, $P_{-n-1}^m(\mu)$, $P_n^{-m}(\mu)$, $P_{-n-1}^{-m}(\mu)$, and by the four corresponding Q functions.

The eight solutions are not independent: they can all be expressed in terms of any two.

The first two solutions are identical (Hobson, *l.c.*, p. 452), *i.e.*

$$P_n^m(\mu) = P_{-n-1}^m(\mu). \tag{14}$$

If this equation is transformed by equation (12) it becomes

$$\frac{1}{\Pi(-m-n-1)}\, e^{+(n+\frac{1}{2})\iota\pi}\, Q_{-m-\frac{1}{2}}^{-n-\frac{1}{2}}(\nu) = \frac{1}{\Pi(-m+n)}\, e^{-(n+\frac{1}{2})\iota\pi}\, Q_{-m-\frac{1}{2}}^{+n+\frac{1}{2}}(\nu),$$

and by writing m for $-(n+\frac{1}{2})$, n for $-(m+\frac{1}{2})$, μ for ν, we find

$$\frac{1}{\Pi(m+n)} e^{-m\iota\pi} Q_n^m(\mu) = \frac{1}{\Pi(n-m)} e^{+m\iota\pi} Q_n^{-m}(\mu), \tag{15}$$

which is Hobson's equation (12).

Again, Hobson's equation (18),

$$P_n^m(\mu) = \frac{e^{-m\iota\pi}}{\pi \cos n\pi} \{ Q_n^m(\mu) \sin(n+m)\pi - Q_{-n-1}^m(\mu) \sin(n-m)\pi \}, \tag{16}$$

transforms by a like process into

$$e^{-m\pi\iota} Q_n^m(\mu) = \frac{\pi}{2 \sin m\pi} \left\{ P_n^m(\mu) - \frac{\Pi(n+m)}{\Pi(n-m)} P_n^{-m}(\mu) \right\}, \tag{17}$$

which is his equation (19).

It should be noted that the transformation of equations (12) and (13) cannot be applied to the relations between Legendre functions of $+\mu$ and $-\mu$, as those equations only hold good when the real parts of μ and ν are positive. As might be anticipated from Fig. 2, the Q's of the parameters $+\mu$ and $-\mu$ are more closely associated than the P's. Actually

$$Q_n^m(-\mu) = -e^{\pm n\pi\iota} Q_n^m(\mu),$$

whilst $$P_n^m(-\mu) = e^{\mp n\pi\iota} P_n^m(\mu) - \frac{2 \sin(n+m)\pi}{\pi} e^{-m\pi\iota} Q_n^m(\mu).$$

8. *Expansion in Series.*

Many ways of expressing Legendre's functions as hypergeometric series are known. It is obvious that when our transformation formulæ are applied to any one of these expansions another will be found.

Two examples will suffice.

Hobson's equation (5),

$$P_n^m(\mu) = \frac{1}{\Pi(-m)} \left(\frac{\mu+1}{\mu-1}\right)^{\frac{1}{2}m} F\left(-n, n+1, 1-m, \frac{1-\mu}{2}\right), \tag{18}$$

is transformed into his equation (57), which may be written

$$e^{-m\pi\iota} Q_n^m(\cosh\alpha) = \frac{\Pi(-\frac{1}{2})\,\Pi(n+m)}{\Pi(n+\frac{1}{2})} \frac{e^{-(n+\frac{1}{2})\alpha}}{\sqrt{(2\sinh\alpha)}} F\left(m+\tfrac{1}{2}, -m+\tfrac{1}{2}, n+\tfrac{3}{2}, \frac{-1}{e^{2\alpha}-1}\right). \tag{19}$$

Again, Hobson's equation (10),

$$e^{-m\pi\iota} Q_n^m(\mu)$$

$$= \frac{\Pi(-\frac{1}{2})\,\Pi(n+m)}{\Pi(n+\frac{1}{2})} \frac{(\mu^2-1)^{\frac{1}{2}m}}{2^{n+1}\mu^{n+m+1}} F\left(\frac{n+m+2}{2},\ \frac{n+m+1}{2},\ n+\tfrac{3}{2},\ \frac{1}{\mu^2}\right), \quad (20)$$

transforms into

$$P_n^m(\mu) = \frac{2^m}{\Pi(-m)} \frac{\mu^{m+n}}{(\mu^2-1)^{\frac{1}{2}m}} F\left(-\frac{n+m-1}{2},\ -\frac{n+m}{2},\ 1-m,\ \frac{\mu^2-1}{\mu^2}\right). \quad (21)$$

By means of the known relations between hypergeometric functions, the last series can be shown to be equivalent to the two series in powers of μ^{-2} in Hobson's equation (22).

9. *Definite Integrals along Real Paths.*

We shall confine our attention to the cases in which $\mu \equiv \cosh\alpha$ is real and greater than unity. The formulæ are written down seriatim for convenience of comparison. Their sources will be mentioned later. There are four groups distinguished by different types of integrand.

Group I.—

$$P_n^{-m}(\cosh\alpha)$$

$$= \sqrt{\left(\frac{2}{\pi}\right)} \frac{1}{\Pi(m-\frac{1}{2})\sinh^m\alpha} \int_0^\alpha (\cosh\alpha-\cosh\phi)^{m-\frac{1}{2}} \cosh(n+\tfrac{1}{2})\phi\, d\phi,$$

where $m+\frac{1}{2}$ is positive; (22)

$$P_n^{-m}(\cosh\alpha) = \sqrt{\left(\frac{2}{\pi}\right)} \frac{\Pi(m-\frac{1}{2})\sinh^m\alpha}{\Pi(m+n)\,\Pi(m-n-1)} \int_0^\infty \frac{\cosh(n+\frac{1}{2})\phi}{(\cosh\alpha+\cosh\phi)^{m+\frac{1}{2}}} d\phi,$$

where $m-n$ and $m+n+1$ are positive; (23)

$$e^{m\pi\iota} Q_n^{-m}(\cosh\alpha)$$

$$= \sqrt{\left(\frac{\pi}{2}\right)} \frac{1}{\Pi(m-\frac{1}{2})\sinh^m\alpha} \int_\alpha^\infty (\cosh\phi-\cosh\alpha)^{m-\frac{1}{2}}\, e^{-(n+\frac{1}{2})\phi}\, d\phi,$$

where $n+\frac{3}{2} > m+\frac{1}{2} > 0$; (24)

$$e^{-m\pi\iota} Q_n^m(\cosh\alpha)$$

$$= \frac{\Pi(m-\frac{1}{2})}{\sqrt{(2\pi)}} \sinh^m\alpha \left\{\int_0^\pi \frac{\cos(n+\frac{1}{2})\psi\, d\psi}{(\cosh\alpha-\cos\psi)^{m+\frac{1}{2}}} - \cos n\pi \int_0^\infty \frac{e^{-(n+\frac{1}{2})\phi}}{(\cosh\alpha+\cosh\phi)^{m+\frac{1}{2}}} d\phi\right\},$$

where $1+n+m > 0$. (25)

Group II.—

$$e^{-m\pi\iota} Q_n^m(\cosh\alpha) = \frac{\Pi(m+n)}{\Pi(n)} \int_0^\beta (\cosh\alpha - \sinh\alpha\cosh\phi)^n \cosh m\phi\, d\phi,$$

where $n+1$ is positive and β is defined by $\cosh\beta = \coth\alpha$; (26)

$$e^{-m\pi\iota} Q_n^m(\cosh\alpha) = \frac{\Pi(n)}{\Pi(n-m)} \int_0^\infty \frac{\cosh m\phi}{(\cosh\alpha + \sinh\alpha\cosh\phi)^{n+1}}\, d\phi,$$

where $n+m+1$, $n-m+1$ are positive; (27)

$$P_n^{-m}(\cosh\alpha) = \frac{1}{\Pi(n)\,\Pi(m-n-1)} \int_\beta^\infty (\cosh\phi\sinh\alpha - \cosh\alpha)^n e^{-m\phi}\, d\phi,$$

where $m > n$, $n+1 > 0$; (28)

$$P_n^{-m}(\cosh\alpha) = \frac{1}{\pi}\,\frac{\Pi(-1-n)}{\Pi(m-1-n)} \left\{ \int_0^\pi (\cosh\alpha - \sinh\alpha\cos\psi)^n \cos m\psi\, d\psi \right.$$
$$\left. - \sin m\pi \int_0^\infty (\cosh\alpha + \sinh\alpha\cosh\psi)^n e^{-m\psi}\, d\psi \right\},$$

where $m > n$. (29)

Group III.—

$$P_n^{-m}(\cosh\alpha) = \frac{\sinh^m\alpha}{2^m\sqrt{\pi}\,.\,\Pi(m-\tfrac{1}{2})} \int_0^\pi (\cosh\alpha + \sinh\alpha\cos u)^{n-m} \sin^{2m} u\, du,$$

where $m+\frac{1}{2}$ is positive; (30)

$$P_n^{-m}(\cosh\alpha)$$
$$= \frac{1}{\sqrt{\pi}}\,\frac{2^m\,\Pi(m-\tfrac{1}{2})}{\Pi(m+n)\,\Pi(m-n-1)\sinh^m\alpha} \int_\beta^\infty \frac{(\cosh u\sinh\alpha - \cosh\alpha)^{n+m}}{\sinh^{2m} u}\, du,$$

where $m-n$ and $m+n+1$ are positive; (31)

$$e^{m\pi\iota} Q_n^{-m}(\cosh\alpha) = \sqrt{\pi}\,\frac{\sinh^m\alpha}{2^m\Pi(m-\tfrac{1}{2})} \int_0^\beta (\cosh\alpha - \sinh\alpha\cosh u)^{n-m} \sinh^{2m} u\, du,$$

where $n+\frac{3}{2} > m+\frac{1}{2} > 0$. (32)

Group IV.—

$$e^{-m\pi\iota}Q_n^m(\cosh\alpha) = \frac{\sinh^m\alpha}{2^{n+1}}\frac{\Pi(m+n)}{\Pi(n)}\int_0^\pi \frac{\sin^{2n+1}u}{(\cosh\alpha+\cos u)^{n+m+1}}\,du,$$

where $n+1$ is positive; (33)

$$e^{-m\pi\iota}Q_n^m(\cosh\alpha) = 2^n\frac{\Pi(n)\sinh^m\alpha}{\Pi(n-m)}\int_\alpha^\infty \frac{(\cosh u-\cosh\alpha)^{n-m}}{\sinh^{2n+1}u}\,du,$$

where $n+m+1$, $n-m+1$ are positive; (34)

$$P_n^{-m}(\cosh\alpha) = \frac{1}{2^n}\frac{\sinh^{-m}\alpha}{\Pi(n)\,\Pi(m-n-1)}\int_0^\alpha \frac{\sinh^{2n+1}u}{(\cosh\alpha-\cosh u)^{n-m+1}}\,du,$$

where $m > n$, $n+1 > 0$. (35)

The formulæ of Group I transform into those of Group II by applying formula (1), and then writing m for $-(n+\frac{1}{2})$, n for $-(m+\frac{1}{2})$, $\cosh\alpha$ for $\coth\alpha$.

The formulæ of Group IV can be derived in the same way from Group III.

Group III is derived from Group I by an algebraic process. For example, in (22), we write

$$e^\phi = \cosh\alpha+\sinh\alpha\cos u, \qquad 2e^\phi(\cosh\alpha-\cosh\phi) = \sinh^2\alpha\sin^2 u,$$

and obtain (30).

The formula (22) is (112) in Hobson's paper, (26) is his (92). (23) is not given by him except for the special case $m = 0$ (111), but its transformation (27) is his (91). (24) is his (118), but he does not give the corresponding equation (28). The equation (29), which has been obtained in a previous paper,* is a generalization of Hobson's (83). The transformation (25) appears to be new.

Of the formulæ in Group III, (30), is Hobson's (67), (32) is his (87). The first formula in Group IV is also given by him (11). No equations corresponding to (25) and (29) occur in the third and fourth groups, as the substitution for e^ϕ in terms of α and u cannot be used with advantage when ϕ is imaginary.

10. *Geometrical Interpretation.*

The Legendre functions with argument greater than unity occur in mathematical physics as toroidal functions.

* *L.c.*, p. 109.

Toroidal coordinates may be defined as follows :—

The circular axis is a circle with centre O and radius a.

The straight axis ZZ' is a straight line through O perpendicular to the plane of the circular axis.

OK is a fixed radius of the circular axis.

P is any point.

The circular axis cuts the plane containing P and OZ in two points A and B, A being on the same side of OZ as P, B on the opposite side.

The toroidal coordinates of P are :—

α given by $PB/PA = e^{\alpha}$,

θ the angle BPA,

ϕ the azimuth of POZ relative to KOZ.

The surfaces $\alpha =$ const. are anchor-rings,

$\theta =$ const. are spheres,

$\phi =$ const. are planes.

With this system of coordinates

$$V = \sqrt{(\cosh\alpha - \cos\theta)}\, P^{m}_{-\frac{1}{2}+n}(\cosh\alpha) \begin{matrix}\cos\\ \sin\end{matrix} n\theta \begin{matrix}\cos\\ \sin\end{matrix} m\phi, \qquad (36)$$

is a solution of Laplace's equation.

Other solutions are obtained by changing the sign of m and by writing Q instead of P, but these solutions are not all independent. [See above, equations (14)–(17).]

Now invert the system with respect to the point K, and take $\sqrt{2}\,a$ as the radius of inversion. The circular axis inverts into a straight line OY through O; the straight axis inverts into a circle equal to the original circular axis and in a plane at right angles thereto. The spheres $\theta =$ const. invert into planes, the planes $\phi =$ const. into spheres. The anchor rings $\alpha =$ const. will invert into anchor rings.

In the diagrams Figs. 4*a*, *b*, *c*, KK' is the common diameter of the two circular axes. Firm lines represent the original straight axis and circular axis and also sections of an anchor-ring by orthogonal planes. Broken lines represent the axes of the inverse system and sections of the corresponding anchor-ring. It will be seen that the anchor-ring and its inverse touch at points in the plane OYZ.

To find the parameter β of one of the inverse rings with reference to the new circular axis, we need only consider the circles in which it is

cut by the plane $\phi = 0$. (Fig. 4a.) These circles are concentric with

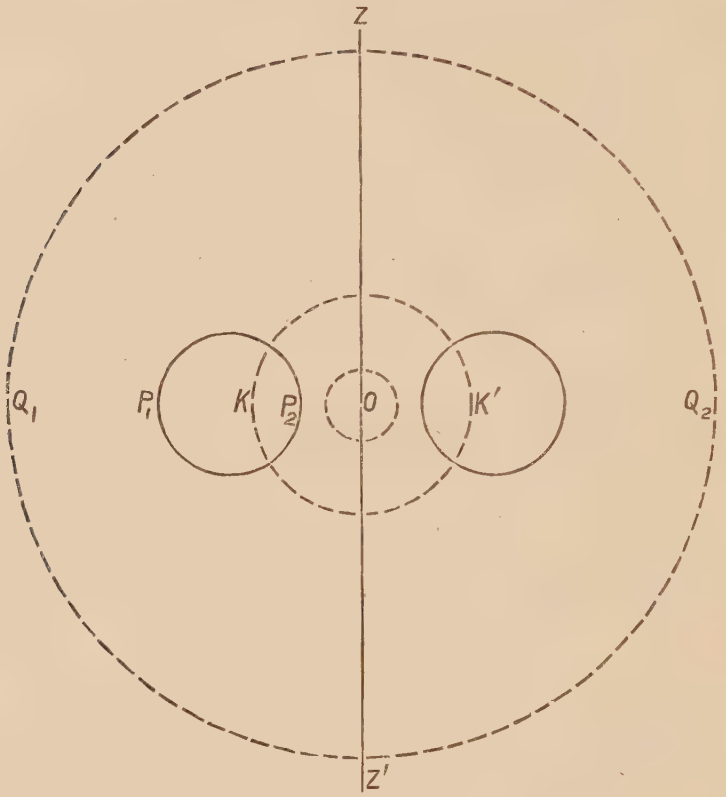

FIG. 4a.

the new circular axis. If OK cut the ring $\alpha = \text{const.}$ at P_1, P_2, and its inverse at Q_1, Q_2, we have

$$KP_1 = \frac{2a}{e^{\alpha}-1}, \qquad KP_2 = \frac{2a}{e^{\alpha}+1},$$

$$KQ_1 = (e^{\alpha}-1)a, \quad KQ_2 = (e^{\alpha}+1)a,$$

so that
$$e^{\beta} \equiv \frac{K'Q_1}{KQ_1} = \frac{KQ_2}{KQ_1} = \frac{e^{\alpha}+1}{e^{\alpha}-1} = \coth\frac{\alpha}{2}.$$

Thus
$$\cosh\beta = \coth\alpha.$$

Accordingly it appears that, if α, θ, ϕ are the toroidal coordinates of a point P with respect to the original circular axis and the plane ZOK, then

β, ϕ, θ are the toroidal coordinates of the inverse point Q with respect to the new circular axis and the plane YOK.

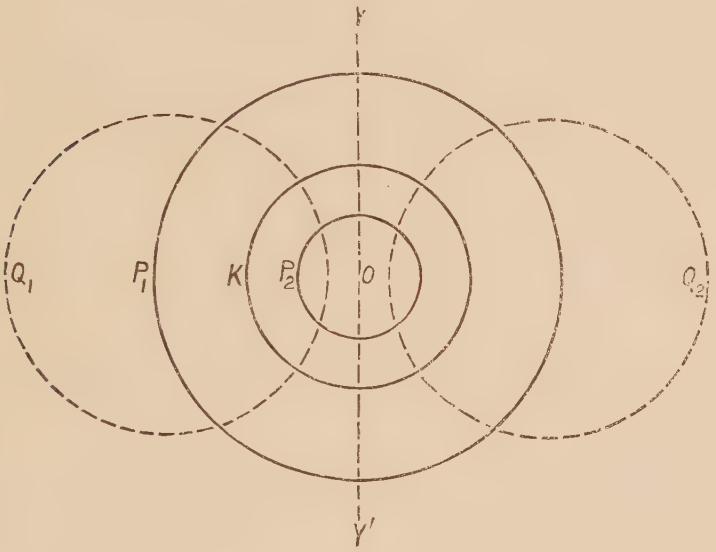

FIG. 4b.

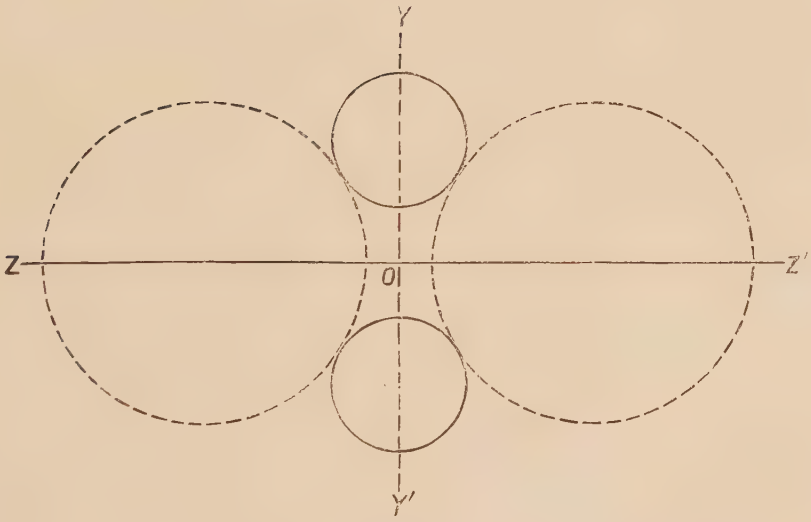

FIG. 4c.

Now the general theory of inversion shows that, if V_P is a solution of

Laplace's equation in the domain of P, then

$$V'_Q \equiv \frac{KP}{\sqrt{2}\,a} \times V_P$$

is a solution in the domain of Q.

It can be shown that

$$KP = \sqrt{2}\,a\,\sqrt{\left(\frac{\cosh\alpha - \sinh\alpha\cos\phi}{\cosh\alpha - \cos\theta}\right)};$$

and therefore, when V_P has the special value given in (36),

$$V'_Q = \sqrt{(\cosh\alpha - \sinh\alpha\cos\phi)}\,P^m_{-\frac{1}{2}+n}(\cosh\alpha)\begin{matrix}\cos\\ \sin\end{matrix} n\theta \begin{matrix}\cos\\ \sin\end{matrix} m\phi.$$

If the transformation of equation (12) is introduced, and $\cosh\beta$ is written for $\coth\alpha$, we get

$$V'_Q = \sqrt{(\cosh\beta - \cos\phi)}\,\sqrt{\frac{2}{\pi}}\,\frac{e^{n i\pi}}{\Pi(-m-n-\frac{1}{2})}\,Q^{-n}_{-\frac{1}{2}-m}(\cosh\beta)\begin{matrix}\cos\\ \sin\end{matrix} n\theta \begin{matrix}\cos\\ \sin\end{matrix} m\phi, \tag{37}$$

as a solution of Laplace's equation in the domain of Q. Thus the formulæ which have been under discussion fall into place as an essential part of the theory of toroidal functions.

11. *An Important Example.*

As an example the formula for the reciprocal of the distance between two points P_1, P_2 with coordinates $(\alpha_1, \theta_1, \phi_1)$, $(\alpha_2, \theta_2, \phi_2)$, such that $\alpha_1 < \alpha_2$, may be considered. In the notation of this paper

$$\frac{1}{P_1P_2} = \frac{(\cosh\alpha_1 - \cos\theta_1)^{\frac{1}{2}}(\cosh\alpha_2 - \cos\theta_2)^{\frac{1}{2}}}{a\pi}$$

$$\times \sum_0^\infty \sum_0^\infty (-)^m \frac{\Pi(n-m-\frac{1}{2})}{\Pi(n+m-\frac{1}{2})} P^m_{n-\frac{1}{2}}(\cosh\alpha_1)\,Q^m_{n-\frac{1}{2}}(\cosh\alpha_2)\,2\cos m(\phi_1-\phi_2)\,2\cos n(\theta_1-\theta_2), \tag{38}$$

where the corresponding 2 in front of a cosine is dropped when m or n is zero.*

When the system is inverted with respect to K, the condition $\alpha_1 < \alpha_2$, taken in conjunction with the relations

$$\sinh\alpha_1 \sinh\beta_1 = 1 = \sinh\alpha_2 \sinh\beta_2,$$

leads to $\beta_1 > \beta_2$. The application of formulæ (12) and (13) leads to the interchange of the P's and the Q's on the right of equation (38), which is therefore transformed into another of the same type.

* Heine, *Kugelfunctionen*, 1881, Vol. 2, p. 287.

ON TWO THEOREMS OF COMBINATORY ANALYSIS AND SOME ALLIED IDENTITIES

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1. In recent developments of Combinatory Analysis certain importance is attached to formulæ established in *Proc. London Math. Soc.*, Ser. 1, Vol. XXV, pp. 328 and 329, in § 5, (1) and (2). As the proofs involved are somewhat intricate, it seems at the present time useful to offer a simpler one, depending directly on well known formulæ.

Using q_n to denote $1-q^n$, $q_n!$ for $q_1 q_2 q_3 \dots q_n$, and (q) for $q_\infty!$, *i.e.*, $\prod_{n=1}^{\infty}(1-q^n)$, it is known that

$$\prod_{n=1}^{\infty}(1+xq^n) = 1+\frac{q}{q_1}x+\frac{q^3}{q_2!}x^2+\frac{q^6}{q_3!}x^3+\dots.$$

Hence $\prod_{n=1}^{\infty}(1+2xq^n\cos\theta+x^2q^{2n})$

$$= \left(1+\frac{q}{q_1}xe^{\theta i}+\frac{q^3}{q_2!}x^2e^{2\theta i}+\dots\right)\left(1+\frac{q}{q_1}xe^{-\theta i}+\frac{q^3}{q_2!}x^2e^{-2\theta i}+\dots\right)$$

$$= 1+\sum_{n=1}^{\infty}\frac{B_n(\theta)}{q_n!}x^n, \qquad (1)$$

according to the notation adopted in Vol. XXIV, p. 352.

It will then be easy to see that

$$B_{2n}(\theta) = q^{n(n-1)}\frac{q_{2n}!}{q_n!\,q_n!}\left\{1+\frac{q_n}{q_{n+1}}2q\cos 2\theta+\frac{q_nq_{n-1}}{q_{n+1}q_{n+2}}2q^4\cos 4\theta+\dots\right\},$$

and

$$B_{2n+1}(\theta)$$

$$= q^{(n+1)^2}\frac{q_{2n+1}!}{q_n!\,q_{n+1}!}\left\{2\cos\theta+\frac{q_n}{q_{n+2}}2q^2\cos 3\theta+\frac{q_nq_{n-1}}{q_{n+2}q_{n+3}}2q^6\cos 5\theta+\dots\right\}.$$

Putting $x = q^{-\frac{1}{2}}$ in (1), it is known by the theory of ϑ-functions that

$$(q)\left\{1+\sum_{n=1}^{\infty}\frac{q^{-\frac{1}{2}n}B_n(\theta)}{q_n!}\right\} = 1+2q^{\frac{1}{2}}\cos\theta+2q^2\cos 2\theta+2q^{\frac{9}{2}}\cos 3\theta+\ldots. \quad (2)$$

Replacing the values of the B's in this identity, it is evident that the coefficients of the several cosines of multiples of θ are equal.

This being so, we may replace the cosines on each side of (2) by any quantities we please consistent with the convergency of the resulting series. Replace

$$2\cos 2\theta,\ 2\cos 4\theta,\ \ldots \text{ by } -(1+q),\ q(1+q^2),\ \ldots,$$

i.e., $$2\cos 2n\theta \text{ by } (-1)^n q^{\frac{1}{2}n(n-1)}(1+q^n),$$

and $$2\cos\theta,\ 2\cos 3\theta,\ \ldots \text{ by } 1-q,\ -(1-q^3),\ \ldots,$$

i.e., $$2\cos(2n+1)\theta \text{ by } (-1)^n q^{\frac{1}{2}n(n-1)}(1-q^{2n+1}),$$

and let β_n denote what $B_n(\theta)$ then becomes.

Then

$$\beta_{2n}\frac{q_n!\,q_n!}{q_{2n}!}q^{-n(n+1)} = 1-q\frac{q_n}{q_{n+1}}(1+q)+q^5\frac{q_nq_{n-1}}{q_{n+1}q_{n+2}}(1+q^2)-\ldots$$

$$= \left(1-q\frac{q_n}{q_{n+1}}\right)-q^2\frac{q_n}{q_{n+1}}\left(1-q^3\frac{q_{n-1}}{q_{n+2}}\right)+\ldots$$

$$= \frac{1}{q_{n+1}}\left\{(1-q)-q^2\frac{q_n}{q_{n+2}}(1-q^3)+\ldots\right\}$$

$$= \frac{1}{q_{n+1}}\beta_{2n+1}\frac{q_n!\,q_{n+1}!}{q_{2n+1}!}q^{-(n+1)^2},$$

by definition of β_{2n+1}, so that

$$\beta_{2n+1} = q^{n+1}(1-q^{2n+1})\beta_{2n}. \quad (3)$$

Moreover

$$(1-q)-q^2\frac{q_n}{q_{n+2}}(1-q^3)+q^7\frac{q_nq_{n-1}}{q_{n+2}q_{n+3}}(1-q^5)-\ldots$$

$$= 1-q\left(1+q\frac{q_n}{q_{n+2}}\right)+q^5\frac{q_n}{q_{n+2}}\left(1+q^2\frac{q_{n-1}}{q_{n+3}}\right)-\ldots$$

$$= 1-q\frac{q_{n+1}}{q_{n+2}}(1+q)+q^5\frac{q_{n+1}q_2}{q_{n+2}q_{n+3}}(1+q^2)-\ldots,$$

whence
$$\beta_{2n+2} = q^{n+1}\frac{1-q^{2n+2}}{1-q^{n+1}}\beta_{2n+1}$$
$$= q^{2n+2}\frac{q_{2n+1}q_{2n+2}}{q_{n+1}}\beta_{2n}.$$

Equating in (2) terms containing cosines of even multiples of θ, and replacing these cosines as above indicated, we have

$$\frac{1}{(q)}(1-q^2-q^3+q^9+q^{11}-\ldots) = 1+\frac{\beta_2 q^{-1}}{q_2!}+\frac{\beta_4 q^{-2}}{q_4!}+\ldots$$
$$= 1+\frac{q}{q_1}+\frac{q^4}{q_2!}+\frac{q^9}{q_3!}+\ldots, \qquad (4)$$

while the series of terms containing odd multiples of θ give

$$\frac{1}{(q)}(1-q-q^4+q^7+q^{13}-\ldots) = \frac{\beta_1 q^{-1}}{q_1}+\frac{\beta_3 q^{-2}}{q_3!}+\ldots$$
$$= \beta_0+\frac{\beta_2}{q_2!}+\frac{\beta_4}{q_4!}+\ldots \text{ [by (3)]}$$
$$= 1+\frac{q^2}{q_1}+\frac{q^6}{q_2!}+\frac{q^{12}}{q_3!}+\ldots. \qquad (5)$$

These results agree with those given in Vol. xxv, pp. 328 and 329, and are the identities referred to in MacMahon's *Combinatory Analysis*, Vol. 2, p. 33.

2. In the memoirs that appeared in Vol. xxv, p. 318 and Vol. xxvi, p. 15, a large number of identities were established, not so much bearing on infinite products as on the connection between series built up of terms consisting of factors of type q_n, and series of the ϑ-function type.

I propose now to add some new theorems and tabulate the formulæ with certain additions and new applications.

The fundamental process of these two memoirs is as follows :—

$A_n(\theta)$ is defined by the identity

$$1+\sum_{n=1}^{\infty}\frac{A_n(\theta)}{q_n!}x^n = \prod_{n=0}^{\infty}(1-2xq^n\cos\theta+x^2q^{2n})^{-1},$$

so that $A_n(\theta) = 2\cos n\theta+\frac{q_n}{q_1}2\cos(n-2)\theta+\frac{q_nq_{n-1}}{q_1q_2}2\cos(n-4)\theta+\ldots,$

or by reversing the order of terms

$$A_{2n}(\theta) = \frac{q_{2n}!}{q_n!\, q_n!}\left(1+\frac{q_n}{q_{n+1}}\,2\cos 2\theta+\frac{q_nq_{n-1}}{q_{n+1}q_{n+2}}\,2\cos 4\theta+\ldots\right),$$

$$A_{2n+1}(\theta) = \frac{q_{2n+1}!}{q_n!\, q_{n+1}!}\left(2\cos\theta+\frac{q_n}{q_{n+2}}\,2\cos 3\theta+\frac{q_nq_{n-1}}{q_{n+2}q_{n+3}}\,2\cos 5\theta+\ldots\right).$$

Now suppose relations are known to exist, viz.,

$$a_0A_0+a_2A_2+\ldots = b_0+2b_2\cos 2\theta+\ldots,$$

and

$$a_1A_1+a_3A_3+\ldots = 2b_1\cos\theta+2b_3\cos 3\theta+\ldots,$$

where A_n is written for $A_n(\theta)$, unless any other argument is involved, then, by certain simple replacements of the cosines, the A's are also found to have simple factorial values.

The notation in general is that when $2\cos r\theta$ is replaced by n_r, then $A_r(\theta)$ is replaced by m_r, giving the identities

$$a_0m_0+a_2m_2+\ldots = b_0+b_2n_2+\ldots,$$

and

$$a_1m_1+a_3m_3+\ldots = b_1n_1+b_3n_3+\ldots.$$

General formulæ for the values of the m_n are usually cumbrous, so that sufficient terms will be given to indicate the law of formation without generalizing. The term m_n consists of binomial factors of the form $1\pm q^r$, and in some cases has a factor q^s, where s is either linear or quadratic in n. The index s will be called the solitary index, and will refer to an a-series, unless specifically stated as referring to a b- or c-series (*v. inf.*).

It may be noticed, though proof is wanting, that if s is other than zero, the b's series can be expressed in terms of differences of the b's, viz.,

$$b_0-b_2,\ b_1-b_3,\ \ldots.$$

These differences may be denoted by

$$c_0,\ c_1,\ \ldots.$$

The n's consist of powers of q only, and generally of not more than two terms. The b-series for the most part lead to ϑ-series of the form

$$1\pm q^p(q^{-m}+q^m)+q^{4p}(q^{-2m}+q^{2m})\pm\ldots,$$

of which it is usually sufficient to write the first three terms. Such a

series is said to be of order p. Thus the series for (q) could be written

$$1-q-q^2+\ldots,$$

being a ϑ-series of order $\frac{3}{2}$.

3. In the following table references will be given to formulæ already proved. Where only the page is given, it refers to Vol. xxv. Notes will follow each group, to explain formulæ to which no reference is appended.

GROUP A, where m_n is a solitary power of q, index of form an^2+bn.

(1) $a_0+a_2+a_4+\ldots = b_0-b_2q-b_4q^2+b_6(q^5+q^7)-b_8q^{12}-b_{10}q^{15}+\ldots,$ p. 325, (7)

(2) $a_1+a_3+a_5+\ldots = b_1-b_3(q+q^2)+b_5q^5+b_7q^7-b_9(q^{12}+q^{15})+\ldots,$ p. 325, (7)

(3) $a_0+a_2q+a_4q^2+\ldots = b_0-b_2-b_4q^4+\ldots$
$= c_0-c_4q^4+c_6q^8-c_{10}q^{20}+c_{12}q^{28}-\ldots,$ p. 325, (9)

(4) $a_1+a_3q+a_5q^2+\ldots = b_1-b_3(1+q^2)+b_5q^2+\ldots$
$= c_1-c_3q^2+c_7q^{10}-c_9q^{16}+c_{13}q^{32}-\ldots,$ p. 325, (8)

(5) $a_0+a_2q+a_4q^4+\ldots = b_0-b_2-b_4q^2+\ldots$
$= c_0-c_4q^2+c_6q^4-c_{10}q^{10}+c_{12}q^{14}-\ldots,$ pp. 338 and 341,

(6) $a_1+a_3q+a_5q^4+\ldots = c_1-(c_3+c_5)q^2+(c_5+c_7)q^4-(c_9+c_{11})q^{10}+\ldots,$

(7) $a_0+a_2+a_4q^2+a_6q^6+\ldots$
$= c_0+c_2-(c_2+c_4)q+(c_6+c_8)q^5-(c_8+c_{10})q^8+\ldots,$ p. 341,

(8) $a_1+a_3q^2+a_5q^6+a_7q^{12}+\ldots$
$= b_1-b_3(1+q)+b_5q+\ldots$
$= c_1-c_3q+c_7q^5-c_9q^8+c_{13}q^{16}-\ldots,$ p. 338.

The indices in the b-series follow those of ϑ-series of order $\frac{3}{2}$, 3, or 6.

The grouping of the powers of q among the b's is remarkable, and, when c-series occur, corresponds to gaps in the continuity of c-suffixes.

A (6) may be proved by § 4 below, but it is not necessary here to work out the proof in full.

GROUP B.—Form of m_r, $q^s q_{2n}!/q_n!$, or $q^s q_{2n+1}!/q_n!$.

(1) $a_0+a_2q_2+a_4q_3q_4+\ldots = b_0-b_2q(1+q)+b_4q^5(1+q^2)-\ldots,$

p. 336, Ex. 1,

(2) $a_0+a_2qq_2+a_4q^2q_3q_4+\ldots = c_0-c_2q^3+c_4q^9-c_6q^{18}+\ldots,$ p. 342,

(3) $a_1q_1+a_3q_2q_3+a_5q_3q_4q_5+\ldots = b_1q_1-b_3q^2q_3+b_5q^7q_5-\ldots,$

(4) $a_1q_1+a_3qq_2q_3+a_5q^2q_3q_4q_5+\ldots = c_1q_1-c_3q^4q_2+c_5q^{11}q_3-c_7q^{21}q_4+\ldots.$

Here the solitary indices in the b-series form hyperarithmetic series of second difference 3.

The formula (3) can be proved by the method of p. 336, and is derived from the b-series for a_1, just as (1) is derived from the b-series for a_0. It may also, together with (4), be found by § 4 below.

GROUP C.—

(1) $a_0+a_2(1+q)+a_4(1+q)(1+q^2)+\ldots$
$= b_0-b_4q^2(1+q^2)+b_8q^{10}(1+q^4)-\ldots,$ p. 341, Ex. 2,

(2) $a_0+a_2q(1+q)+a_4q^2(1+q)(1+q^2)+\ldots$
$= c_0+c_2q^2-c_4q^4-c_6q^{10}+c_8q^{14}+\ldots,$

(3) $a_1+a_3(1+q)+a_5(1+q)(1+q^2)+\ldots$
$= b_1-b_3q^2-b_5q^4+b_7q^{10}+\ldots,$

(4) $a_1+a_3q(1+q)+a_5q^2(1+q)(1+q^2)+\ldots$
$= c_1-c_5q^6+c_9q^{18}-c_{13}q^{36}+\ldots,$

(5) $a_0+a_2(1+q)+a_4q(1+q)(1+q^2)+\ldots$
$= c_0+c_2-(c_4+c_6)q^2+(c_6+c_8)q^6-\ldots,$ p. 340,

(6) $a_1+a_3(1+q)+a_5q(1+q)(1+q^2)+\ldots$
$= b_1-b_5+b_9q^2-b_{13}q^6+\ldots-b_3q^2+b_7q^6-b_{11}q^{12}+\ldots,$

(7) $a_1+a_3q(1+q)+a_5q^3(1+q)(1+q^2)+\ldots$
$= c_1-c_5q^2+c_9q^6-c_{13}q^{12}+\ldots.$

In (5), (6), (7) the solitary indices in the a-series are triangular numbers.

(2) is obtained from subtracting B (3) from B (2), and applying the method of p. 340, § 10.

(4) comes from B (2) by the same method, but (3), (6), (7) will be explained in § 4 below.

GROUP D.

(1) $a_0+a_2(1+2q^{\frac{1}{2}}\cos 2\phi+q)+a_4(1+2q^{\frac{1}{2}}\cos 2\phi+q)(1+2q^{\frac{3}{2}}\cos 2\phi+q^3)+\ldots$
$= b_0+2b_2q^{\frac{1}{2}}\cos 2\phi+2b_4q^2\cos 4\phi+\ldots$, Vol. XXVI, p. 27,

(2) $a_1.2\cos\phi+a_3.2\cos\phi(1+2q\cos 2\phi+q^2)$
$+a_5.2\cos\phi(1+2q\cos 2\phi+q^2)(1+2q^2\cos 2\phi+q^4)+\ldots$
$= 2b_1\cos\phi+2b_3q\cos 3\phi+2b_5q^3\cos 5\phi+\ldots$,

whence, by putting $1+2q^{\frac{1}{2}}\cos 2\phi+q=0$ in (1),

(3) $a_0 = b_0-b_2(1+q)+b_4q(1+q^2)-\ldots$ p. 322, (12)
$= c_0-c_2q+c_4q^3-c_6q^6+\ldots$,

(4) $a_1q_1 = c_1q_1-c_3qq_2+c_5q^3q_3-\ldots$, p. 322. (13)

Moreover, in (1), $\phi=0$ gives a result equivalent to Ex. 3, p. 339, and $2\phi=0$ gives the same result as Ex. 4, p. 341.

In (2), $\phi=0$ and $\phi=\frac{\pi}{4}$ give

(5) $a_1+a_3(1+q)^2+a_5(1+q)^2(1+q^2)^2+\ldots = b_1+b_3q+b_5q^3+\ldots$,

(6) $a_1+a_3(1+q^2)+a_5(1+q^2)(1+q^4)+\ldots = b_1-b_3q-b_5q^3+b_7q^6+\ldots$.

In (1) let $e^{2\phi i}=iq^{\frac{1}{2}}$, and we get, by equating real and imaginary parts,

(7) $a_0+2a_2+2a_4(1+q^2)+2a_6(1+q^2)(1+q^4)+\ldots$
$= b_0+b_2-(b_2+b_4)q+(b_4+b_6)q^3-\ldots$,

(8) $a_0+2a_2q+2a_4q^2(1+q^2)+2a_6q^3(1+q^2)(1+q^4)+\ldots$
$= c_0+c_2q-c_4q^3-c_6q^6+\ldots$.

The last combined with (3) gives

(9) $a_0+a_2q+a_4q^2(1+q^2)+\ldots = c_0-c_6q^6+c_8q^{10}-c_{14}q^{28}+\ldots$,

the solitary indices in the c-series being non-successive triangular numbers.

Formula (2) is not proved in Vol. xxvi, along with (1), but the method of proof is analogous and need not be expounded here in full.

It may further be noticed in connection with (3) and (4) that b-series for the separate a's, viz., a_2, a_3, &c., may be obtained successively from (1) and (2), but the results are all practically given in Vol. xxv, p. 322 (11). They are apparently of no special importance in the present paper.

GROUP E.—

(1) $a_0+a_2q_1+a_4q_1q_3+a_6q_1q_3q_5+\ldots$

$$= b_0-2b_2q+2b_4q^4-\ldots,$$ p. 336, Ex. 2,

(2) $a_0-a_2q_1+a_4q_1q_3-\ldots$

$$= b_0-2b_2+2b_4-\ldots,$$,,

(3) $a_1q_1+a_3q_1q_3+a_5q_1q_3q_5+\ldots$

$$= b_1q_1-b_3qq_3+b_5q^4q_5-b_7q^9q_7+\ldots,$$ by § 4,

(4) $a_0+a_2qq_1+a_4q^2q_1q_3+\ldots$

$$= c_0-c_2q^2+c_4q^6-c_6q^{12}+\ldots,$$ p. 337, Ex. 4,

(5) $a_0-a_2q^{-1}q_1+a_4q^{-2}q_1q_3-\ldots$

$$= b_0-b_2q^{-1}(1+q^2)+b_4q^{-2}(1+q^4)-\ldots,$$,,

(6) $a_1q_1+a_3qq_1q_3+a_5q^2q_1q_3q_5-\ldots$

$$= \frac{1}{1+q}(b_1q_2-b_3q_6+b_5q^2q_{10}-b_7q^6q_{14}+\ldots),$$ by § 4,

(7) $a_1q_1-a_3q^{-1}q_1q_3+a_5q^{-2}q_1q_3q_5-\ldots$

$$= b_1q_1-b_3q^{-1}q_3+b_5q^{-2}q_5-\ldots,$$ by § 4.

GROUP F.—Solitary indices in A.P.

(1) $a_0+a_2(1+q^{\frac{1}{2}})(1+q)+\ldots$

$$= b_0+b_2q^{\frac{1}{2}}(1+q)+b_4q^3(1+q^2)+\ldots,$$ p. 337, Ex. 3,

(2) $a_1(1+q^{\frac{1}{2}})+a_3(1+q^{\frac{1}{2}})(1+q)(1+q^2)+\ldots$

$$= b_1(1+q^{\frac{1}{2}})+b_3q^{\frac{3}{2}}(1+q^{\frac{3}{2}})+\ldots,$$,,

(3) $a_0+a_2q^{-\frac{1}{2}}(1+q^{\frac{1}{2}})(1+q)+\ldots$

$$= b_0+b_2q^{-\frac{1}{2}}(1+q)+b_4q^{-1}(1+q^2)+\ldots, \text{ p. 337, Ex. 3,}$$

(4) $a_1(1+q^{\frac{1}{2}})+a_3q^{-\frac{1}{2}}(1+q^{\frac{1}{2}})(1+q)(1+q^{\frac{3}{2}})+\ldots$

$$= b_1(1+q^{\frac{1}{2}})+b_3q^{-\frac{1}{2}}(1+q^{\frac{3}{2}})+\ldots, \quad \text{„} \quad .$$

GROUP G.—

(1) $a_0+a_2(1-q^{\frac{1}{2}})+a_4(1-q^{\frac{1}{2}})(1-q^{\frac{3}{2}})+\ldots$

$$= b_0-b_2q^{\frac{1}{2}}(1+q^{\frac{1}{2}})+b_4q^{\frac{3}{2}}(1+q)-\ldots, \quad \text{p. 339, Ex. 4,}$$

(2) $a_1(1-q^{\frac{1}{2}})+a_3(1-q^{\frac{1}{2}})(1-q^{\frac{3}{2}})+\ldots$

$$= b_1(1-q^{\frac{1}{2}})-b_3q(1-q^{\frac{3}{2}})+b_5q^{\frac{7}{2}}(1-q^{\frac{5}{2}})-\ldots,$$

(3) $a_0+a_2q(1-q^{\frac{1}{2}})+a_4q^2(1-q^{\frac{1}{2}})(1-q^{\frac{3}{2}})+\ldots$

$$= c_0-c_2q^3+c_4q^{\frac{9}{2}}-c_6q^9+\ldots, \quad \text{p. 342,}$$

(4) $a_0-a_2q^{\frac{1}{2}}(1-q^{\frac{1}{2}})+a_4q^2(1-q^{\frac{1}{2}})(1-q^{\frac{3}{2}})-\ldots$

$$= c_0-c_2q^{\frac{1}{2}}+c_4q^{\frac{3}{2}}-c_6q^3-\ldots, \quad \text{p. 339, Ex. 2,}$$

(5) $a_1(1-q^{\frac{1}{2}})-a_3q^{\frac{1}{2}}(1-q^{\frac{1}{2}})(1-q^{\frac{3}{2}})+a_5q^2(1-q^{\frac{1}{2}})(1-q^{\frac{3}{2}})(1-q^{\frac{5}{2}})+\ldots$

$$= b_1(1-q^{\frac{1}{2}})-b_3(1-q^{\frac{3}{2}})+b_5q^{\frac{1}{2}}(1-q^{\frac{5}{2}})-\ldots.$$

Here (2) may be got from B (3), and (5) from D (4), by the method of p. 338.

In (4) and (5) the solitary indices are half the natural squares.

4. It is easy to show (Vol. XXIV, p. 344) that

$$2\cos\theta . A_{n-1}(\theta) = A_n(\theta)+(1-q^{n-1})A_{n-2}(\theta).$$

Hence, if $a_0A_0+a_1A_1+\ldots = b_0+2b_1\cos\theta+\ldots,$

we find, by multiplying by $2\cos\theta$, that

$$a_0A_1+a_1(A_2+q_1A_0)+a_2(A_3+q_2A_1)+\ldots = 2b_0\cos\theta+2b_1(1+\cos 2\theta)+\ldots,$$

so that, given a relation

$$a_0m_0+a_1m_1+\ldots = b_0n_0+b_1n_1+\ldots,$$

we also have

$$a_0m_1+a_1(m_2+q_1m_0)+a_2(m_3+q_2m_1)+\ldots$$

$$= b_0n_1+b_1(2n_0+n_2)+b_2(n_1+n_3)+b_3(n_2+n_4)+\ldots,$$

while $$a_1m_1+a_3m_3+\ldots = b_1n_1+b_3n_3+\ldots,$$

implies $$a_0m_1+a_2(m_3+q_2m_1)+\ldots = b_0n_1+b_2(n_1+n_3)+\ldots.$$

Reference has been made to this section in many of the formulæ of § 3. The deductions are always sufficiently easy not to require detailed proof.

5. Having obtained the above identities in a- and b-series, it remains to find to what identities connecting A-series with cosine-series we may apply them.

We should notice that when

$$a_0A_0+a_1A_1+a_2A_2+\ldots = b_0+2b_1\cos\theta+2b_2\cos 2\theta+\ldots,$$

the latter series may also be written

$$\frac{1}{\sin\theta}(c_0\sin\theta+c_1\sin 2\theta+c_2\sin 3\theta+\ldots),$$

as this form sometimes facilitates calculation for such relations as are simplified by use of the c's.

The most general formulæ are those given in Vol. XXVI, pp. 19, 20, 22, 26, which are

$$1+(u-q^{\frac12})(v-q^{\frac12})\frac{A_2}{q_2!}+(u-q^{\frac12})(u-q^{\frac32})(v-q^{\frac12})(v-q^{\frac32})\frac{A_4}{q_4!}+\ldots$$

$$=\frac{(uq^{\frac12})(vq^{\frac12})}{(q)(uv)}\left\{1+2\cos 2\theta\frac{(u-q^{\frac12})(v-q^{\frac12})}{(1-uq^{\frac12})(1-vq^{\frac12})}\right.$$

$$\left.+2\cos 4\theta\frac{(u-q^{\frac12})(u-q^{\frac32})(v-q^{\frac12})(v-q^{\frac32})}{(1-uq^{\frac12})(1-uq^{\frac32})(1-vq^{\frac12})(1-vq^{\frac32})}+\ldots\right\}$$

$$=\frac{(uq^{\frac32})(vq^{\frac32})}{(q)(uvq)}\frac{1}{\sin\theta}\left\{q_1\sin\theta+\frac{(u-q^{\frac12})(v-q^{\frac12})}{(1-uq^{\frac32})(1-vq^{\frac32})}q_3\sin 3\theta+\ldots\right\}, \quad (1)$$

where (u) denotes $\prod_{n=0}^{\infty}(1-uq^n)$; while

$$\frac{A_1}{q_1}+(u-q)(v-q)\frac{A_3}{q_3!}+(u-q)(u-q^2)(v-q)(v-q^2)\frac{A_5}{q_5!}+\ldots$$

$$=\frac{(uq)(vq)}{(q)(uv)}\left\{2\cos\theta+\frac{(u-q)(v-q)}{(1-uq)(1-vq)}2\cos 3\theta\right.$$

$$\left.+\frac{(u-q)(u-q^2)(v-q)(v-q^2)}{(1-uq)(1-uq^2)(1-vq)(1-vq^2)}2\cos 5\theta+\ldots\right\}. \quad (2)$$

Moreover, on p. 26, Vol. XXVI,

$$1+\frac{A_1(\theta)\,A_1(\phi)}{q_1!}+\frac{A_2(\theta)\,A_2(\phi)}{q_2!}+\ldots=\frac{1}{\sin\theta}\,\frac{\sin\theta\sin\phi+\sin 2\theta\sin 2\phi+\ldots}{\sin\phi-q\sin 3\phi+q^3\sin 5\phi-\ldots},\qquad(3)$$

which, as it stands, gives divergent series, but may be applied to any cases in the groups where the solitary indices are positive and not zero, the right-hand series being then expressed in terms of the c's.

We then get in some cases new A-series in ϕ, but the right-hand sides are complicated by the ϑ-series in the denominator of (3), and lead to quotients of series which are not reducible to simple Fourier forms, unless the zeroes of the denominator are all zeroes of the numerator. No application will be made of this formula in the present paper.

6. We shall now apply the preceding formulæ to special values of u and v.

From § 5 (1), we get:—

$u=v=0$;

$$1+q\,\frac{A_2}{q_2!}+q^4\,\frac{A_4}{q_4!}+q^9\,\frac{A_6}{q_6!}+\ldots=\frac{1}{(q)}(1+2q\cos 2\theta+2q^4\cos 4\theta+\ldots):\quad(1)$$

$u=1$, $v=0$, and changing $q^{\frac12}$ into $-q^{\frac12}$,

$$1+q^{\frac12}(1+q^{\frac12})\,\frac{A_2}{q_2!}+q^2(1+q^{\frac12})(1+q^{\frac13})\,\frac{A_4}{q_4!}+\ldots$$

$$=\frac{(-q^{\frac12})}{(q)}(1+2q^{\frac12}\cos 2\theta+2q^2\cos 4\theta+\ldots):\quad(2)$$

$u=-q^{\frac12}$, $v=0$;

$$1+2q\,\frac{A_2}{q_2!}+2q^3(1+q)\,\frac{A_4}{q_4!}+\ldots$$

$$=\frac{(-q)}{(q)}\left(1+\frac{4q}{1+q}\cos 2\theta+\frac{4q^3}{1+q^2}\cos 4\theta+\ldots\right):\quad(3)$$

$u=-1$, $v=1$;

$$1-\frac{A_2}{q_2}+\frac{A_4}{q_2q_4}-\ldots=\frac{(q^{\frac12})(-q^{\frac12})}{2(q)(-q)}(1-2\cos 2\theta+2\cos 4\theta-\ldots):\quad(4)$$

$u=-q^{\frac12}$, $v=1$, and changing $q^{\frac12}$ into $-q^{\frac12}$;

$$1+\frac{2q^{\frac{1}{2}}}{1-q^{\frac{1}{2}}}\cdot\frac{A_2}{q_2}+\frac{2q}{(1-q^{\frac{1}{2}})(1-q)(1-q^{\frac{3}{2}})}\cdot\frac{A_4}{q_4}+\ldots$$

$$=\frac{(-q)(-q^{\frac{1}{2}})}{(q)(q^{\frac{1}{2}})}\left(1+\frac{4q^{\frac{1}{2}}}{1+q}\cos 2\theta+\frac{4q}{1+q^2}\cos 4\theta+\ldots\right): \quad (5)$$

$u=-v=iq^{\frac{1}{2}}$;

$$1+2q\frac{A_2}{q_2!}+2q^2(1+q^2)\frac{A_4}{q_4!}+\ldots$$

$$=\prod_{n=1}^{\infty}\left(\frac{1+q^{2n}}{1-q^{2n}}\right)\left(1+\frac{4q}{1+q^2}\cos 2\theta+\frac{4q^2}{1+q^4}\cos 4\theta+\ldots\right): \quad (6)$$

Of these it may be noticed that the cosine series are all forms familiar in elliptic function theory, except in (3), which is a series of a type produced by integrating through a real period the dn-series in (5) multiplied by the ϑ-series

$$1+2q^{\frac{1}{2}}\cos 2\theta\cos 2\phi+2q^2\cos 4\theta\cos 4\phi+\ldots.$$

From § 5 (2), we get :—

$u=0,\ v=0$;

$$\frac{A_1}{q_1}+q^2\frac{A_3}{q_3!}+q^6\frac{A_5}{q_5!}+\ldots=\frac{1}{(q)}(2\cos\theta+2q^2\cos 3\theta+2q^6\cos 5\theta+\ldots): \quad (7)$$

$u=-q^{\frac{1}{2}},\ v=0$;

$$\frac{A_1}{q_1}+q^{\frac{3}{2}}(1+q^{\frac{1}{2}})\frac{A_3}{q_3!}+q^4(1+q^{\frac{1}{2}})(1+q^{\frac{3}{2}})\frac{A_5}{q_5!}+\ldots$$

$$=\frac{(-q^{\frac{1}{2}})}{(q)}\left(\frac{2\cos\theta}{1+q^{\frac{1}{2}}}+\frac{2q^{\frac{3}{2}}\cos 3\theta}{1+q^{\frac{3}{2}}}+\frac{2q^4\cos 5\theta}{1+q^{\frac{5}{2}}}+\ldots\right): \quad (8)$$

$u=-q^{-\frac{1}{2}},\ v=0$;

$$(1+q^{\frac{1}{2}})\frac{A_1}{q_1}+q^{\frac{1}{2}}(1+q^{\frac{1}{2}})(1+q^{\frac{3}{2}})\frac{A_3}{q_3!}+q^2(1+q^{\frac{1}{2}})(1+q^{\frac{3}{2}})(1+q^{\frac{5}{2}})\frac{A_5}{q_5!}+\ldots$$

$$=\frac{(-q^{\frac{1}{2}})}{(q)}\{2(1+q^{\frac{1}{2}})\cos\theta+2q^{\frac{1}{2}}(1+q^{\frac{3}{2}})\cos 3\theta+2q^2(1+q^{\frac{5}{2}})\cos 5\theta+\ldots\}: (9)$$

$u=-1,\ v=0$;

$$\frac{A_1}{q_1}+q(1+q)\frac{A_3}{q_3!}+q^3(1+q)(1+q^2)\frac{A_5}{q_5!}+\ldots$$

$$=\frac{(-q)}{(q)}(2\cos\theta+2q\cos 3\theta+2q^3\cos 5\theta+\ldots): \qquad (10)$$

$u=1,\ v=0$;

$$\frac{A_1}{q_1}-qq_1\frac{A_3}{q_3!}+q^3q_1q_2\frac{A_5}{q_5!}-\cdots$$

$$=2\cos\theta-2q\cos 3\theta+2q^3\cos 5\theta-\ldots: \qquad (11)$$

$u=q^{\frac{1}{2}},\ v=-q^{\frac{1}{2}}$;

$$\frac{A_1}{q_1}-qq_1\frac{A_3}{q_3!}+q^2q_1q_3\frac{A_5}{q_5!}-\ldots$$

$$=\frac{(q^{\frac{1}{2}})(-q^{\frac{1}{2}})}{(q)(-q)}\left(\frac{2\cos\theta}{1-q}-\frac{2q\cos 3\theta}{1-q^3}+\frac{2q^2\cos 5\theta}{1-q^5}-\ldots\right): \qquad (12)$$

$u=v=-q^{\frac{1}{2}}$;

$$\frac{A_1}{q_1}+q(1+q^{\frac{1}{2}})^2\frac{A_3}{q_3!}+\ldots$$

$$=\frac{(-q^{\frac{1}{2}})^2}{(-q)^2}\left\{\frac{2\cos\theta}{(1+q^{\frac{1}{2}})^2}+\frac{2q\cos 3\theta}{(1+q^{\frac{3}{2}})^2}+\ldots\right\}: \qquad (13)$$

$u=-1,\ v=-q^{\frac{1}{2}}$;

$$\frac{A_1}{q_1}+q^{\frac{1}{2}}(1+q^{\frac{1}{2}})(1+q)\frac{A_3}{q_3!}+\ldots$$

$$=\frac{(-q)(-q^{\frac{1}{2}})}{(q)(q^{\frac{1}{2}})}\left(\frac{2\cos\theta}{1+q^{\frac{1}{2}}}+\frac{2q^{\frac{1}{2}}\cos 3\theta}{1+q^{\frac{3}{2}}}+\ldots\right): \qquad (14)$$

$u=1,\ v=-q^{\frac{1}{2}}$;

$$\frac{A_1}{q_1}-q^{\frac{1}{2}}(1+q^{\frac{1}{2}})(1-q)\frac{A_3}{q_3!}+\ldots$$

$$=\frac{2\cos\theta}{1+q^{\frac{1}{2}}}-\frac{2q^{\frac{1}{2}}\cos 3\theta}{1+q^{\frac{3}{2}}}+\frac{2q\cos 5\theta}{1+q^{\frac{5}{2}}}-\ldots: \qquad (15)$$

$u=1,\ v=-1$;

$$\frac{A_1}{q_1}-\frac{A_3}{q_1q_3}+\frac{A_5}{q_1q_3q_5}-\ldots$$

$$=\tfrac{1}{2}(2\cos\theta-2\cos 3\theta+2\cos 5\theta-\ldots): \tag{16}$$

$u=-1,\ v=-1;$

$$\frac{A_1}{q_1}+(1+q)^2\frac{A_3}{q_3!}+(1+q)^2(1+q^2)^2\frac{A_5}{q_5!}+\ldots$$

$$=\frac{(-q)^2}{(q)^2}\cdot\frac{1}{\sin\theta}\left(\frac{1-q}{1+q}\sin 2\theta+\frac{1-q^2}{1+q^2}\sin 4\theta+\ldots\right): \tag{17}$$

$u=i,\ v=-i;$

$$\frac{A_1}{q_1}+(1+q^2)\frac{A_3}{q_3!}+(1+q^2)(1+q^4)\frac{A_5}{q_5!}+\ldots$$

$$=\tfrac{1}{2}\prod_{n=1}^{\infty}\left(\frac{1+q^{2n}}{1-q^{2n}}\right)(2\cos\theta+2\cos 3\theta+\ldots): \tag{18}$$

$u=-v=iq^{\frac{1}{2}};$

$$\frac{A_1}{q_1}+q(1+q)\frac{A_3}{q_3!}+q^2(1+q)(1+q^3)\frac{A_5}{q_5!}+\ldots$$

$$=\prod_{n=1}^{\infty}\left(\frac{1+q^{2n-1}}{1-q^{2n}}\right)\left(\frac{2\cos\theta}{1+q}+\frac{2q\cos 3\theta}{1+q^3}+\ldots\right). \tag{19}$$

Of these formulæ (7) to (19), most give elliptic function series.

(8), like (3), depends on a definite integral, but the most remarkable are (11) and (15), which may be called false elliptic series, as being series which would be elliptic if all the minus signs were changed to plus signs.

(4), (16), (17), (18) are non-convergent, and can only be applied to the a- and c-relations of § 3.

(11) leads to very remarkable applications, as in all cases the b-series become false ϑ-series, as will be seen in § 9 below. (15) is also a false cn-series, but I have not investigated its applications.

Among the above formulæ we have not included that in Vol. xxv, p. 318, by which the identities in § 1 were first established. It is not one that lends itself to any other simple results, as the coefficients of the A's are all infinite series.

Note.—Inadvertently I have omitted a formula corresponding to (11), which is of importance later.

From § 5 (1), when $u = q^{-\frac{1}{2}}$, $v = 0$, we have

$$1-q_1\frac{A_2}{q_2!}+qq_1q_2\frac{A_4}{q_4!}-q^3q_3!\frac{A_6}{q_6!}+\ldots$$

$$=\frac{1}{\sin\theta}(q_1\sin\theta-q_3\sin 3\theta+qq_5\sin 5\theta-\ldots)$$

$$=-2q_1\cos 2\theta+2qq_2\cos 4\theta-2q^3q_3\cos 6\theta+\ldots. \quad (20)$$

Similarly, when $u = -q^{-\frac{1}{2}}$, $v = 0$,

$$1+(1+q)\frac{A_2}{q_2!}+q(1+q)(1+q^2)\frac{A_4}{q_4!}+\ldots$$

$$=\frac{(-q)}{(q)}\{2+2(1+q)\cos 2\theta+2q(1+q^2)\cos 4\theta+\ldots\}. \quad (21)$$

7. *Applications of formulæ in* § 3 *to* § 6.

As we have upwards of 40 a- and b-formulæ in § 3 applicable to 18 special formulæ in § 6, we may obtain a very large number of identities of the type under consideration, but many results of the possible 720 will be identical, probably essentially so.

A very limited selection will therefore be made, and will be grouped according to similarity of order in the resulting ϑ-series.

Firstly, we may prove the combinatory formulæ of § 1, by taking B (1) with (1), where, as in all following deductions, B (2) refers to Group B in § 3, and (1) refers to § 6.

This gives § 1 (4).

Similarly, B (2) with (1), gives § 1 (5).

B (3) and B (4) with (7) give the same.

Again, C (1) with (1), C (3) with (7), C (4) with (7), give

$$(q)\left\{1+\frac{q(1+q)}{q_2!}+\frac{q^4(1+q)(1+q^2)}{q_4!}+\ldots\right\}=1-q^6-q^8+\ldots,$$

i.e.

$$\left.\begin{aligned}(q)\left\{1+\frac{q}{q_1\cdot q_1}+\frac{q^4}{q_1q_3\cdot q_1q_2}+\frac{q^9}{q_1q_3q_5\cdot q_3!}+\ldots\right\}&=1-q^6-q^8+\ldots,\\ (q)\left(\frac{1}{q_1}+\frac{q^2}{q_1q_3\cdot q_1}+\frac{q^6}{q_1q_3q_5\cdot q_2!}+\ldots\right)&=1-q^4-q^{10}+\ldots,\\ (q)\left(\frac{1}{q_1}+\frac{q^3}{q_1q_3\cdot q_1}+\frac{q^8}{q_1q_3q_5\cdot q_2!}+\ldots\right)&=1-q^2-q^{12}+\ldots,\end{aligned}\right\} \quad (1)$$

three ϑ-functions of order 7.

Again C (6) and C (7) with (7) give

$$\left.\begin{aligned}(q)\left(\frac{1}{q_1}+\frac{q^2}{q_1q_3\cdot q_1}+\frac{q^7}{q_1q_3q_5\cdot q_2!}+\ldots\right)&=1-q^4-q^6+\ldots,\\(q)\left(\frac{1}{q_1}+\frac{q^3}{q_1q_3\cdot q_1}+\frac{q^9}{q_1q_3q_5\cdot q_2!}+\ldots\right)&=1-q^2-q^8+\ldots,\end{aligned}\right\}. \qquad (2)$$

two ϑ-functions of order 5, while C (5) with (1) also gives

$$(q)\left(1+\frac{q}{q_1\cdot q_1}+\frac{q^5}{q_1q_3\cdot q_2!}+\ldots\right)=1-q^4-q^6+\ldots$$

Again, C (1) with (2), after changing q into q^2, and C (4) with (8) give

$$\left.\begin{aligned}1+\frac{q}{q_2!}+\frac{q^4}{q_4!}+\ldots&=\prod_{n=1}^{\infty}\left(\frac{1+q^{2n-1}}{1-q^{2n}}\right)(1-q^8-q^{12}+\ldots)\\\frac{1}{q_1}+\frac{q^3}{q_3!}+\frac{q^8}{q_5!}+\ldots&=\prod_{n=1}^{\infty}\left(\frac{1+q^{2n-1}}{1-q^{2n}}\right)(1-q^4-q^{16}+\ldots)\end{aligned}\right\}, \qquad (3)$$

for which formula see also Vol. xxv, p. 329, § 6.

These are ϑ-functions of order 10.

Again, C (3) with (10), C (4) with (10), C (1) with (3), give

$$\left.\begin{aligned}\frac{1}{q_1}+q\,\frac{(1+q)^2}{q_3!}+q^3\frac{(1+q)^2(1+q^2)^2}{q_5!}+\ldots&=\frac{(-q)}{(q)}(1-q^3-q^7+\ldots)\\\frac{1}{q_1}+q^2\frac{(1+q)^2}{q_3!}+q^5\frac{(1+q)^2(1+q^2)^2}{q_5!}+\ldots&=\frac{(-q)}{(q)}(1-q-q^9+\ldots)\\1+2q\,\frac{1+q}{q_2!}+2q^3\frac{(1+q)^2(1+q^3)}{q_4!}+\ldots&=\frac{(-q)}{(q)}(1-2q^5+2q^{20}+\ldots)\end{aligned}\right\}, \qquad (4)$$

i.e., ϑ-functions of order 5.

Again, G (4) with (1), and G (5) with (7), give

$$\left.\begin{aligned}(q)\left\{1-q^{\frac{3}{2}}\frac{1-q^{\frac{1}{2}}}{q_2!}+q^6\frac{(1-q^{\frac{1}{2}})(1-q^{\frac{3}{2}})}{q_4!}-\ldots\right\}&\\=1-q-q^{\frac{3}{2}}+\ldots\ \text{(p. 339)}&\\(q)\left\{\frac{1-q^{\frac{1}{2}}}{q_1}-q^{\frac{5}{2}}\frac{(1-q^{\frac{1}{2}})(1-q^{\frac{3}{2}})}{q_3!}+q^8\frac{(1-q^{\frac{1}{2}})(1-q^{\frac{3}{2}})(1-q^{\frac{5}{2}})}{q_5!}-\ldots\right\}&\\=1-q^{\frac{1}{2}}-q^2+\ldots&\end{aligned}\right\}, \qquad (5)$$

i.e., ϑ-functions of order $\frac{5}{4}$.

Again, G (1) with (1), G (2) with (7), and G (3) with (1), give

$$\left.\begin{aligned}
(q)\left\{1+q\,\frac{1-q^{\frac{1}{2}}}{q_2!}+q^4\frac{(1-q^{\frac{1}{2}})(1-q^{\frac{2}{2}})}{q_4!}+\ldots\right\} &= 1-q^{\frac{3}{2}}-q^2+\ldots \quad \text{(p. 339)}\\
(q)\left\{\frac{1-q^{\frac{1}{2}}}{q_1}+q^2\frac{(1-q^{\frac{1}{2}})(1-q^{\frac{3}{2}})}{q_3!}+\ldots\right\} &= 1-q^{\frac{1}{2}}-q^3+\ldots\\
(q)\left\{1+q^2\frac{1-q^{\frac{1}{2}}}{q_2!}+q^6\frac{(1-q^{\frac{1}{2}})(1-q^{\frac{3}{2}})}{q_4!}+\ldots\right\} &= 1-q\ -q^{\frac{5}{2}}+\ldots
\end{aligned}\right\}, \tag{6}$$

i.e., ϑ-functions of order $\frac{7}{4}$.

8. In this section we shall deduce results from Group A, which, by virtue of the peculiar character in the coefficients of every third b or c, give rise to ϑ-functions whose orders are multiples of 3. Cases have already been given in Vol. xxv, p. 333, where actual connections between ϑ-series of orders $\frac{5}{2}$ and 15 are obtained.

The following combinations, A (3) with (21), A (2) and A (4) with (10), give

$$\left.\begin{aligned}
\frac{(q)}{(-q)}\left(1+\frac{q}{q_1\cdot q_1}+\frac{q^3}{q_1q_3\cdot q_2!}+\frac{q^6}{q_1q_3q_5\cdot q_3!}+\ldots\right)&\\
= 1+q^{10}+q^{11}+\ldots-q\,(1+q^4+q^{17}+\ldots)&\\
\frac{(q)}{(-q)}\left(\frac{1}{q_1}+\frac{q}{q_1q_3\cdot q_1}+\frac{q^3}{q_1q_3q_5\cdot q_2!}+\ldots\right)&\\
= 1+q^8+q^{13}+\ldots-q^2(1+q+q^{20}+\ldots)&\\
\frac{(q)}{(-q)}\left(\frac{1}{q_1}+\frac{q^2}{q_1q_3\cdot q_1}+\frac{q^5}{q_1q_3q_5\cdot q_2!}+\ldots\right)&\\
= 1+q^5+q^{16}+\ldots-q(1+q^2+q^{17}+\ldots)&
\end{aligned}\right\}, \tag{1}$$

i.e., pairs of ϑ-functions of order $\frac{21}{2}$.

It may be observed that the leading indices in the six ϑ-series account for all the numbers less than and prime to 21, so that there are not likely to be any further analogous identities existent. The same remark applies to the following, with regard to numbers less than and prime to 15.

Again, A (5) with (21), A (8) with (10), give

$$\left.\begin{aligned}&\frac{(q)}{(-q)}\left(1+\frac{q}{q_1\cdot q_1}+\frac{q^5}{q_1q_3\cdot q_2!}+\frac{q^{12}}{q_1q_3q_5\cdot q_3!}+\ldots\right)\\&\qquad=1+q^7+q^8+\ldots-q(1+q^2+q^{13}+\ldots)\\&\frac{(q)}{(-q)}\left(\frac{1}{q_1}+\frac{q^2}{q_1q_3\cdot q_1}+\frac{q^7}{q_1q_3q_5\cdot q_2!}+\ldots\right)\\&\qquad=1+q^4+q^{11}+\ldots-q(1+q+q^{14}+\ldots)\end{aligned}\right\},\quad(12)$$

which are of order $\frac{15}{2}$.

Comparing (2) of this series of results with § 7 (2), we see that

$$1+q^7+q^8+\ldots-q(1+q^2+q^{13}+\ldots)$$
$$=(1-q^4-q^6+\ldots)\div(1+q)(1+q^2)(1+q^3)\ldots$$

and $1+q^4+q^{11}+\ldots-q(1+q+q^{14}+\ldots)$
$$=(1-q^2-q^8+\ldots)\div(1+q)(1+q^2)(1+q^3)\ldots,$$

two identities connecting $\mathfrak{H}$-series of orders 5 and $\frac{15}{2}$.

Similar identities do not occur for $\mathfrak{H}$-series of order 7 and $\frac{21}{2}$, for the solitary indices in the present (1), and those in § 7 (1), are not identical.

9. *Identities involving false $\mathfrak{H}$-series.*

We shall now apply the formula of the groups in § 5 to the two formulæ § 6 (20) and (11), but it will not be necessary to specify in each case which of these two is being used, as the former applies to formulæ involving even suffixes in a and b, while the latter involves only odd. It may be observed that no infinite products occur as factors in the following identities.

From

$$\left.\begin{aligned}&\text{A (2)},\quad\frac{1}{q_1}-\frac{q}{q_2q_3}+\frac{q^3}{q_3q_4q_5}-\ldots\\&\qquad=1+q^8-q^{13}-\ldots+q^2(1+q-q^{20}-\ldots)\\&\text{A (3)},\quad1-\frac{q}{q_2}+\frac{q^3}{q_3q_4}-\frac{q^6}{q_4q_5q_6}+\ldots\\&\qquad=1+q^{10}-q^{11}-q^{41}+q^{43}+\ldots-q(1+q^4-q^{17}-q^{29}+q^{55}+\ldots)\\&\text{A (4)},\quad\frac{1}{q_1}-\frac{q^2}{q_2q_3}+\frac{q^5}{q_3q_4q_5}-\ldots\\&\qquad=1+q^5-q^{16}-q^{31}+\ldots+q(1+q^2-q^{19}-q^{25}+\ldots)\end{aligned}\right\},\quad(1)$$

false $\mathfrak{H}$-series of order $\frac{21}{2}$.

$$\left.\begin{aligned}
&\text{A (5)},\quad 1-\frac{q}{q_2}+\frac{q^5}{q_3q_4}-\frac{q^{12}}{q_4q_5q_6}+\ldots \\
&\qquad = 1-q^7+q^8+q^{29}-q^{31}-\ldots-q(1+q^2-q^{13}-q^{19}+q^{41}+\ldots) \\
&\text{A (6)},\quad \frac{1}{q_1}-\frac{q^2}{q_2q_3}+\frac{q^7}{q_3q_4q_5}-\ldots \\
&\qquad = 1+q^7-q^8-q^{29}+q^{31}+\ldots+q(1+q^2-q^{13}-q^{19}+\ldots) \\
&\qquad \text{(so that the sum of the two series is 2)} \\
&\text{A (8)},\quad \frac{1}{q_1}-\frac{q^3}{q_2q_3}+\frac{q^9}{q_3q_4q_5}-\ldots \\
&\qquad = 1+q^4-q^{11}-q^{23}+q^{37}+\ldots+q(1+q-q^{14}-q^{17}+\ldots)
\end{aligned}\right\}, \tag{2}$$

false ϑ-series of order $\frac{15}{2}$.

The B-group gives unimportant results.

$$\left.\begin{aligned}
&\text{From C (3)},\quad \frac{1}{q_1}-\frac{q}{q_1q_3}+\frac{q^3}{q_1q_3q_5}-\ldots = 1+q^3-q^7-q^{16}+q^{24}+\ldots \\
&\text{C (4)},\quad \frac{1}{q_1}-\frac{q^2}{q_1q_3}+\frac{q^5}{q_1q_3q_5}-\ldots = 1+q-q^9-q^{12}+q^{20}+\ldots
\end{aligned}\right\}, \tag{3}$$

false ϑ-series of order 5,

$$\text{C (6)},\quad \frac{1}{q_1}-\frac{q}{q_1q_3}+\frac{q^4}{q_1q_3q_5}-\ldots = 1,$$

$$\text{C (7)},\quad \frac{1}{q_1}-\frac{q^2}{q_1q_3}+\frac{q^6}{q_1q_3q_5}-\ldots = 1+q-q^5-q^8+q^{16}+\ldots, \tag{4}$$

a false ϑ-series of order 3.

See also the effect of c-formulæ in Group A on § 6 (16).

$$\left.\begin{aligned}
&\text{From D (5)},\quad \frac{1}{q_1}-q\frac{1+q}{q_1q_3}+q^3\frac{(1+q)(1+q^2)}{q_1q_3q_5}-\ldots = 1-q^2+q^6-q^{12}+\ldots \\
&\text{D (6)},\quad \frac{1}{q_1}-q\frac{1+q^2}{q_2q_3}+q^3\frac{(1+q^2)(1+q^4)}{q_3q_4q_5}-\ldots = 1+q^2-q^6-q^{12}+\ldots
\end{aligned}\right\}, \tag{5}$$

false ϑ-series of order 4.

$$\text{From E (3)},\quad 1-\frac{q}{1+q}+\frac{q^3}{(1+q)(1+q^2)}-\ldots = 1-q+q^2-q^5+q^7-\ldots, \tag{6}$$

a false ϑ-series of order $\frac{3}{2}$, which will be examined more generally in § 10.

Group F also gives results included in § 10.

From G (2), after changing $q^{\frac{1}{2}}$ into q,

$$\left.\begin{aligned}\frac{1}{1+q}-\frac{q^2}{(1+q)(1+q^2)(1+q^3)}+\frac{q^6}{(1+q)(1+q^2)(1+q^3)(1+q^4)(1+q^5)}-\ldots\\ =1-q+q^4-q^7+q^{13}-\ldots\end{aligned}\right\}$$

G (3),

$$\left.\begin{aligned}1-\frac{q^2}{(1+q)(1+q^2)}+\frac{q^6}{(1+q)(1+q^2)(1+q^3)(1+q^4)}-\ldots\\ =1-q^2+q^3-q^9+q^{11}-\ldots\end{aligned}\right\},$$

false ϑ-series of order $\frac{5}{2}$. (7)

10. Some of the preceding identities may be obtained independently by the following method.

Let $$f(a, c, x) \equiv 1+\frac{1-a}{1-c}x+\frac{(1-a)(1-aq)}{(1-c)(1-cq)}x^2+\ldots,$$

the Heinean series $\phi(a, q, c, q, x)$. Then

$$f(a, c, x) = 1+\frac{1-a}{1-c}xf(aq, cq, x),$$

$$f(a, c, x)-af(a, c, xq) = (1-a)f(aq, c, x),$$

$$(1-c)f(a, c, x) = f(a, cq, x)-cf(a, cq, xq),$$

from which three relations we may deduce

$$f(a, c, x) = \frac{1-ax}{1-x}+x\frac{(1-a)(c-axq)}{(1-c)(1-x)}f(aq, cq, xq)$$

$$= \frac{1-ax}{1-x}+x\frac{(1-a)(c-axq)}{(1-c)(1-x)(1-xq)}(1-axq^2)$$

$$+x^2q^2\frac{(1-a)(1-aq)(c-axq)(c-axq^2)}{(1-c)(1-cq)(1-x)(1-xq)(1-xq^2)}(1-axq^4)$$

$$+\ldots, \qquad (1)$$

the solitary indices in q being of form $n(n-1)$.

If now we put $x = 0$, $a = \infty$, $ax = y^2q$, and $c = yq$, and write $f(y)$ for $f(a, c, x)$, so that $f(aq, cq, xq)$ would be $f(yq)$, then (1) becomes

$$f(y) = 1-y^2q-y^3q^2f(yq), \qquad (2)$$

$$i.e.,\ f(y) = 1-\frac{y^2q}{1-yq}+\frac{y^4q^3}{(1-yq)(1-yq^2)}-\frac{y^6q^6}{(1-yq)(1-yq^2)(1-yq^3)}+\dots$$

$$= 1-y^2q-y^3q^2f(yq)$$

$$= 1-y^2q-y^3q^2+y^5q^5+y^6q^7-y^8q^{12}-y^9q^{15}+\dots, \qquad (3)$$

where the powers of q follow those of the ϑ-series $1-q-q^2+\dots$, and the coefficient of y^{3n+1} is always zero.

This is a generalization of the identity

$$1-\frac{q}{1-q}+\frac{q^3}{q_2!}-\dots = (q) = 1-q-q^2+q^5+q^7-\dots.$$

Again, in (1), if $c=0$, $a=x$, and $F(x)$ be written for $f(a, c, x)$, we get $F(x) = 1+x-x^3qF(xq) = 1+x-x^3q-x^4q^2+x^6q^5+x^7q^7-\dots,$

so that we have finally

$$(1-x)+x(1-x)(1-xq)+x^2(1-x)(1-xq)(1-xq^2)+\dots$$

$$= 1-x^2q-x^3q^2+x^5q^5+x^6q^7-\dots. \qquad (4)$$

If, in (3), we put $y=q^{-\frac{1}{2}}$, and subtract each side from unity and change $q^{\frac{1}{2}}$ into q, we get

$$\frac{1}{q_1}-\frac{q^2}{q_1q_3}+\frac{q^6}{q_1q_3q_5}-\dots = 1+q-q^5-q^8+q^{16}+\dots,$$

which is the same as § 9 (4).

If $y=-1$, we have § 9 (6).

We may, in conclusion, point out that when $c=aq$, (1) leads to an interesting identity

$$\frac{1}{1-a}+\frac{x}{1-aq}+\frac{x^2}{1-aq^2}+\dots$$

$$= \frac{1-ax}{(1-a)(1-x)}+axq\frac{1-axq^2}{(1-aq)(1-xq)}+a^2x^2q^4\frac{1-axq^4}{(1-aq^2)(1-xq^2)}+\dots,$$

the solitary indices in q being the natural squares.

When $x=a=q$, this gives the transformation for Lambert's series

$$\frac{x}{1-x}+\frac{x^2}{1-x^2}+\frac{x^3}{1-x^3}+\dots = x\frac{1+x}{1-x}+x^4\frac{1+x^2}{1-x^2}+x^9\frac{1+x^3}{1-x^3}+\dots.$$

CORRIGENDA IN PREVIOUS PAPERS RELATING TO THE PRESENT.

Vol. XXIV, p. 171 : omit third comma in (2), giving bx instead of b, x.
p. 173, l. 2 : insert x as a factor in the right-hand side.
p. 175, in formula (12) : write 4) θ for 4θ.
p. 177, l. 2 : + for − in coefficient of $2x^2q^{2n}\cos 2\theta$.

Vol. XXV, p. 322, in formula (11) : in coefficient of $b_4\lambda^2$, write q_3 for q_2.
p. 323, last identity : coefficient of $-q^{10}$ should be $1-2q^6+2q^{13}-2q^{21}+\ldots$, and that of $+q^{14}$, $1-2q^7+2q^{15}-\ldots$.
p. 325, formula (7) : coefficient of b_8 should be $-q^{12}$.
p. 331, formula (8) : for $\psi(1-q)$, write $\psi(-q)$.
p. 338, l. 5 from the bottom : for a write a.
p. 339, in Example 4 : for § 9 write § 8.
p. 340 : for $A_2(2\theta, q^2)$, $A_4(2\theta, q^2)$, write $A_1(2\theta, q^2)$, $A_2(2\theta, q^2)$.
p. 341, Example 3 : for b_0+qb_2-, &c., write

$$b_0-qb_2-b_4+q(1+q^4)b_6-q^8b_8-\ldots .$$

Example 4 : change + to − in denominator of coefficient of μ^2 in the first series.

Vol. XXVI, p. 15, formula (2) : write $1/P(x)$ for $P(x)$. Formula (8) : write f_{2r-1} for f_r, and correct accordingly on p. 21. Formulæ on pp. 27, 28 will now be correct.
p. 18, l. 3 : for μ_1 write μ_2.
p. 23, l. 4 : for $(1-cq)^{r-1}!$ write $c_r!$ or $(1-cq^{r-1})!$.
p. 24, last series : for $A_3(\theta)$, $A_5(\theta)$, write $A_3(\phi)$, $A_5(\phi)$.
p. 26 : for $\sin 6\phi$ write $\sin 8\phi$.
p. 28, l. 17 : insert $(1-q^{\frac{3}{2}})$ in coefficient of f_1f_3.
p. 29, l. 5 : write $1-vq^{\frac{1}{2}}$ for $v-q^{\frac{1}{2}}$ in the denominator of the second term of the series.

ON THE INTERNAL STRUCTURE OF A SET OF POINTS IN SPACE OF ANY NUMBER OF DIMENSIONS

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1. In attempting to pass from the Theory of Functions of a Single Variable to that of two or more variables, one is frequently stopped by finding that some result, hitherto utilisable, depends on the peculiar constitution of linear space, and cannot be at once extended to the plane. It may be that no such extension is possible. On the other hand, one may, by a happy intuition, arrive at a result in the plane which proves itself to be the required extension, and opens out the whole series of theorems which we had in mind. It is such generalisations which are interesting in themselves as throwing light on the essential differences between the straight line and the plane as range of the independent variable, and they usually show us the direction to be followed in extending our theory to higher space. The same phenomena may, however, occur in passing from any space to one of more dimensions.

This has been the case with those parts of the Theory of Functions which deal with the limits of a function as we approach definite points. The first of these theorems stated* that, except at a countable set of points, the upper and lower limits of $f(x+h)$ and $f(x-h)$, h being positive and approaching zero, are the same whatever function $f(x)$ may be. More generally we may say that *the ultimate internal structure of the most general function $f(x)$ is symmetrical on the right and left* except at a countable set of points, since† the result above stated holds not only

* W. H. Young, "On the Distinction of Right and Left at Points of Discontinuity", 1907, British Association (Leicester), *Quart. Journ. of Pure and Applied Math.*, Vol. 39, pp. 67–83.

† W. H. Young, "Sulle due funzioni a piu valori costituite dai limiti d'una funzione d'una variabile a destra e a sinistra di ciascun punto", 1908, *Rend. d. R. Acc. dei Lincei*, t. 17, pp. 582–587.

for the upper and lower limits, but for the whole set of limits on the right and on the left of the point x.

No one who was in the least familiar with the vagaries of functions of a single variable could have expected such remarkable symmetry to manifest itself in the structure of a perfectly general function of a single variable when examined, so to speak, microscopically. But, this result found, it would be even more incredible if it were only in the case of a single variable that such regularity existed. It was not, however, at once evident what form of symmetry was to be expected, nor what form of exceptional set would be obtained.

Application of the method which had been successful on the straight line led indeed to a generalisation,* interesting in itself, and fraught with curious consequences, but which did not impress itself as final. The exceptional set found was of the first category, but as one of us wrote in August, 1909, "Various considerations suggest the possibility that this set of the first category has content zero."† The attention of the writer was, however, then diverted to the theory of Fourier Series, and to the development of this theory in various directions.

In the course of investigations in this subject various possibilities in the employment of several variables suggested itself, and in a paper "On Multiple Fourier Series", published in January, 1912, results were obtained which opened up a vista of further consequences, several of which have been attained.‡ More recently in the extension of the Theory of Integration,§ previously developed by the same author, to integration with respect to functions of any number of variables, the conviction was strengthened that the step from one dimension to higher space was one which could not prove any longer insurmountable, even in cases where previous efforts had failed.

We were thus led to take up afresh the question of the structure of a function of any number of dimensions. After one or two partially successful attempts, it appeared that the complete solution of the problem re-

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* W. H. Young, "On the Discontinuities of a Function of One or More Real Variables", 1909, *Proc. London Math. Soc.*, Ser. 2, Vol. 8, pp. 118–124.

† W. H. Young, "A Note on Monotone Functions", 1909, *Quarterly Journal*, Vol. 41, pp. 79–86.

‡ W. H. Young, "On Multiple Fourier Series", 1912, *Proc. London Math. Soc.*, Ser. 2, Vol. 11, pp. 135–184. G. H. Hardy and J. E. Littlewood, "Contributions to the Arithmetic Theory of Series", *ibid.*, pp. 415–472.

§ See the various papers published on this subject by W. H. Young in the *Proceedings* of the Royal and London Mathematical Societies, in particular "On Multiple Integration by Parts and the Second Theorem of the Mean", *Proc. London Math. Soc.*, 1916, Ser. 2, Vol. 16, pp. 273–293, and "On Multiple Integrals", *Proc. Roy. Soc.*, 1916 (A), Vol. 93, pp. 28–41.

quired new theorems in the Theory of Sets of Points in the plane or higher space, which are not only of interest and importance in themselves, but lead actually to the confirmation of the surmise above referred to, that the set of the first category of exceptional singularities of a function of any number of variables has content zero.

In the present communication we propose to give an account of these theorems in the Theory of Sets of Points.

2. It falls naturally into two parts : the first is descriptive ; the second metrical.

On the straight line the matter was very simple. The distinction between limiting points of a set which were limiting points on both sides, and those which were limiting points on one side only, though fundamental, was obvious, and it was easily seen that the latter limiting points form a countable set. It was the noticing of this simple distinction which led to the result in the Theory of Functions of a Single Variable above referred to, as well as to others obtained later.

When we pass to higher space, we have to determine, what is not at first sight at all obvious, the analogue of a limiting point on one side only ; having decided this question, we have to solve the problem of the nature of the sub-set formed by the points of a set which have the analogous property.

It appears that the generalisation required for the purpose in hand—one which throws great light on the structure of sets of points in general—is based primarily on the distinguishing of (i) *points of the set such that, inside every possible triangle having such a point as vertex, there are points of the set*, and (ii) *points which do not possess this property*. The points of the second class are those at which the set has *a triangular void*, and therefore a whole set of triangular voids. A corresponding distinction may be made for space of any number of dimensions, the appropriate element taking the place of the triangle. The second fundamental note in our generalisation is given by the theorem that *the points at which a set has a triangular void are not only nowhere dense, but lie on a countably infinite aggregate of non-oscillatory curves, constituting accordingly a set of content zero.*

To obtain this theorem, we discuss first of all sets of a particular type, in which at every point there is a void, the form of which is taken to be that of a rectangle, with sides horizontal and vertical, having the point in question for one of its corners. Such a void we may call *a flag*. It is then shown that *the presence of a certain flag* (*e.g.*, *that above and on the*

right) *at every point of a set, entails the presence of the opposite flag* (*e.g.*, *that below and on the left*). Thus the sets in question fall into two classes, those which have two flags and those which have four.

We then have the following theorems :—

(1) *A set with four flags lies on a countable set of parallels to the axes.*

(2) *A set with two flags lies on a countable aggregate of non-oscillatory curves ; in both cases, of course, the set is dense nowhere.*

The flags here do not include their bounding lines. If one pair of these is included, we have in place of (1) the simple but interesting theorem that *the points which are limiting points of a set on the right* (*left*) *but not on the left* (*right*) *lie on a countable set of vertical lines, and those which are limits above* (*below*) *but not below* (*above*) *on a countable set of horizontal lines. If both pairs of bounding lines are included, the set with four flags is of course countable.*

From the main fundamental theorem we obtain the following :—

A plane set which can be divided up into a countably infinite aggregate of sets, each possessing at each of its points a triangular void, is a set of the first category, of content zero, and lies on a countable aggregate of non-oscillatory curves.

The classification of the points of a plane set which we have made is particularly interesting when the set is a region. The "natural" or "normal" boundary of a region, consisting of all the points of the region which are the vertices of triangles whose internal points all belong to the region (and are accordingly voids of the complementary set), is, by our main theorem, a set of content zero, and lies on a countable aggregate of non-oscillatory curves ; the remaining boundary points of the region, which may form a set of positive content, are what we have called *stickle-points of the region.**

The paper concludes with the metrical theorem that *any plane set may be divided up into a sub-set S′ of content zero, and a sub-set S″ having the property that, inside every triangle having inside it or on its boundary a point of S″, the points of S″ form a set of positive content.*

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* This result was communicated lately to the Académie des Sciences.

The generalisation of these theorems for space of three or more dimensions is immediate.

3. Every new result in the Theory of Plane Sets of Points may, in the present state of the science, be not only of value in itself, but also in the light it may throw on the theory generally. We are thus led to examine anew the definition of a plane curve as hitherto formulated, and more especially of a Jordan curve. This latter is defined as a set of points which is in (1, 1) continuous correspondence with the straight line or circle. Even a Jordan curve has been shewn* with this definition to be capable of possessing positive (plane) content. The theorem of the present paper last quoted suggests that the term "curve" might be with propriety restricted to such Jordan loci as are composed exclusively of points belonging to the first of our two classes; or, more precisely, we shall require from a curve that, when divided up into a sufficiently large number of portions, or even a countably infinite number of such portions, the set of points constituted by any one of the portions is a set, which, relatively to itself, includes only points having triangular voids. This would be a step further in the direction initiated by denying the term "curve" to what was originally referred to as *the Peano curve filling up an area*. A plane curve, properly so called, would then always be of zero plane content.

4. To simplify the preliminary treatment of the problem, let us take a pair of rectangular axes through our point P in fixed directions, which we will call horizontal and vertical, and consider the structure of our plane set S at P with reference to the quadrants so determined. We will number these clockwise beginning with the top right-hand quadrant (Fig. 1). The quadrants are to be considered as completely open, that is to say, neither of the axes is included. Corresponding to these we shall have what we call *the first, second, third, and fourth neighbourhoods of P*. The boundary of such a neighbourhood is, of course, quite immaterial, except in so far as it must contain P and touch the axes at P. For convenience we may take it to be a rectangle with horizontal and vertical sides and P as appropriate corner.

FIG. 1.

* Young's *Theory of Sets of Points*, pp. 244 *et seq.*

These neighbourhoods, unless otherwise stated, are completely open rectangles, and are indicated in the figures by simple shading. Sometimes, however, it is convenient to include one or other of the axes; in this case they become half closed (vertically, or horizontally) or closed. In the figures this will be indicated by double shading in the neighbourhood of the included axis. (Fig. 2.)

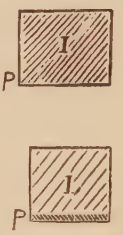

Fig. 2.

The structure of any plane set S containing P will be said to be *biaxially symmetrical at* P, if in each of the quadrants there are points of S as near P as we please. There will then be different kinds of biaxial symmetry according as the axes are not, or are, included in the various quadrants. Similarly if there are points of S as near P as we please, on each side of the vertical (horizontal) through P, the structure of the set at P will be uniaxially symmetrical about the vertical (horizontal), or we may say that at P the set is *symmetrical as regards right and left* (*upper and lower*).

What now are the forms of plane sets containing no limiting points of biaxial or uniaxial symmetry with respect to our system of axes? This problem is a generalisation of that already solved for the straight line. What are the forms of sets devoid of limiting points on both sides? That the answer cannot still be that such sets are countable must be at once evident; indeed, the points of a straight line, circle, ellipse, or other simple plane curve satisfy the requirements, and are of potency c. The answer cannot therefore depend only on potency. It will be seen from the theorems obtained that the set has, in all the different cases which occur, one of the most important descriptive properties of a countable set, or of the curves just cited, namely, *it lies on a countable aggregate of non-oscillatory curves.* It is therefore a set of the first category with the most important metrical property of a countable set; *it has, in fact, content zero.*

5. If S is a plane set devoid of points of biaxial symmetry, there must, at each of its points P, be at least one of the first four neighbourhoods empty of points of the set. The points of the set may accordingly be classified according to the first of these neighbourhoods which is empty. Denoting these sub-sets by S_1, S_2, S_3, and S_4, the set S_1 is such that each point P of it is the left-hand bottom corner of a rectangle, whose sides are horizontal and vertical, containing no point of S_1 in its interior. A similar statement holds for S_2, S_3, and S_4, the appropriate corner being substituted in each case. It is therefore only necessary to discuss a set of the type S_1.

The following Lemmas render the problem still simpler.

LEMMA.—*If S is a plane set of points, such that each of its points P is the left-hand bottom corner of a rectangle, whose sides are horizontal and vertical, containing no point of S in its interior, then S is the sum of a finite or countably infinite set of sub-sets,*

$$S = S_1 + S_2 + \ldots + S_i + \ldots,$$

where the points of each sub-set S_i are such that corresponding to each there is a square with that point as left-hand bottom corner, having horizontal and vertical sides of length 2^{-i} and containing no point of S in its interior.

It is evident that, since there is such a rectangle as is described in the enunciation corresponding to each point P of the set S, there are any number of such rectangles; in fact any smaller rectangle with horizontal and vertical sides, having P for left-hand bottom corner, will contain no point of S in its interior. Each rectangle will determine uniquely a positive integer i, namely the least integer such that 2^{-i} is less than either of the sides. The square with horizontal and vertical sides of length 2^{-i}, having P for left-hand bottom corner, will then contain no point of the set S in its interior. Moreover there will be one definite such square, corresponding to the least such value of i determined by all the rectangles associated with the point P, which will contain all the others, so that these latter may be suppressed. We have thus associated with each point P of the set S a definite integer i, and we may group together all those points which have the same associated integer, into a sub-set S_i. These sub-sets have no common point, and they contain all the points of S, which proves the theorem.

A similar lemma holds, of course, when we take any other corner instead of the left-hand bottom corner.

LEMMA 2.—*If S_i is a plane set of points, such that each of its points is the left-hand bottom corner of a square, with horizontal and vertical sides of length 2^{-i}, containing no point of S in its interior, then each point of S_i is also the right-hand top corner of another such square.*

If the former square be taken to include one or both of its bounding lines through the point considered, the same bounding line is included in the latter square.

Each limiting point of S_i is also the left-hand bottom corner of a

similar square, and also the right-hand top corner of such a square, not necessarily including, however, either bounding line.

Let L be any point of S_i, and d_{++} the square with L as left-hand bottom corner, with horizontal and vertical sides of length 2^{-i}, and d_{--} the similar square with L as right-hand top corner (Fig. 3), these squares being considered to be either (i) completely open, or (ii) both to include the vertical, or (iii) the horizontal, or (iv) both bounding lines through P.

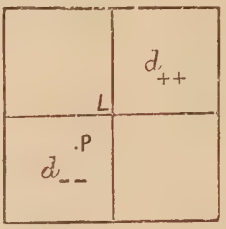

Fig. 3.

Let P be any point of d_{--}. In case (i) P is an internal point of the square d_{--}, in the remaining cases P is either an internal point, or a point on an included boundary, but P is not identical with L. Then a similar square with P as left-hand bottom corner will have L for one of its points; in case (i) L will be an internal point; in the remaining cases L will be an internal point, or lie on an included boundary. This square, therefore, is not one of the given squares, and therefore P is not a point of S_i. Thus d_{--} contains no point of S_i inside it or on any included boundary. This proves the Lemma in so far as the points of S_i are concerned.

Next, let L be a limiting point of S_i, and d_{++} and d_{--} be completely open squares. A square d with horizontal and vertical sides of length 2^{-i}, having its left-hand bottom corner in d_{--}, or, what is the same thing, its right-hand top corner in d_{++}, will contain L as internal point, and therefore will contain points of S_i, since L is a limiting point of S_i. Such a square d is therefore not empty of points of s_i, and therefore, by what has been proved, its right-hand top, and left-hand bottom, points do not belong to S_i. Thus d_{++} and d_{--} are empty of points of S_i. This proves the Lemma.

6. In this Lemma we may, of course, interchange *right* and *left*, *upper* and *lower*. Thus the Lemma tells us that a point of a set of the type contemplated must either have one pair or two pairs of vertically opposite squares with the point as common corner, free of points of the set. Thus there are three types of points which come in: (1) that contemplated explicitly in the Lemma; (2) that got by exchanging right and left; (3) that in which all the four completely open, half-open, or closed squares round the point as common corner are free of points of the set. In the last case mentioned, when the squares are closed, the point is an isolated point of the set; in all the other cases it is a limiting point of the set.

The types of limiting point which such a set S may have are accord-

ingly the following :—(1) *limiting points above on the left, below on the right ;* (2) *limiting points above on the right, below on the left ;* (3*a*) *horizontal-vertical limiting points,* such that all the points of S_i in the neighbourhood lie on the axial cross through the limiting point considered ; (3*b*) *horizontal,* and (3*c*) *vertical limiting points,* in which the points of S in the neighbourhood lie on one or other of the axes through the limiting point in question.

Each of the classes (1) and (2) of limiting points includes all the three remaining classes, and class (3*a*) includes (3*b*) and (3*c*). It will be seen by the following theorems, in which, of course, right and left may be interchanged, that *a set which only has limiting points of these types is capable of being arranged on a countably infinite set of monotone curves, while, if the two first types are absent, these curves take the simpler form of horizontal and vertical stretches.*

It is the latter case, as simplest, which we deal with first. By the preceding Lemma 2, any two other adjacent squares with P as corner may be substituted for those given in the enunciation.

THEOREM 1.—*If S is a plane set of points, each point P of which is at the same time the top left-hand corner of a rectangle, whose sides are horizontal and vertical, containing no point of S in its interior, and also the bottom left-hand corner of such a rectangle, then the set S lies on a countable set of horizontal-vertical cross-lines.*

By Lemma 1, we may divide up the set S into the sets S_1, S_2, ..., and it is evident that if, for each integer i, the theorem is true when for S we substitute S_i, the theorem is true as it stands. We may therefore assume that the given rectangles are all squares of the same size, the length of the side being 2^i (Fig. 4).

It then follows, by Lemma 2, that P is also the right-hand top, and the right-hand bottom, corner of such squares containing no points of S_i. That is, P is the centre of a square, consisting of these four squares, containing no points of S_i except on the pair of horizontal-vertical lines through its centre P (Fig. 5).

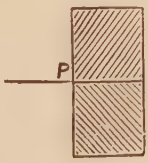

FIG. 4.

FIG. 5.

Now let us divide up the plane, chessboard fashion, into equal squares, or chequers, with horizontal and vertical sides of length 2^{-i}, each chequer being supposed to contain its left-hand and bottom sides, and its left-hand bottom corner. There is a countably infinite set of such chequers, no two chequers have a common point, and the points of all the chequers are the points of the whole plane. Thus the set S_i is the sum of its sub-sets in the chequers.

Now, if in any chequer there is a point P of the set S_i, the whole chequer is covered over by the square above referred to, four times its size (Fig. 6), containing no point of S_i except on the pair of horizontal

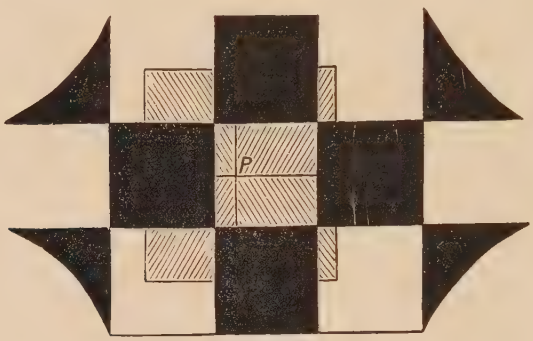

Fig. 6.

vertical lines through its centre P. Thus all the points of S_i in the chequer lie on this pair of horizontal-vertical lines.

This proves the theorem.

Cor. 1.—*If either of the two rectangles mentioned in the enunciation includes the horizontal (vertical) side through P (so that on this side also there is no other point of the set S), then the set S lies on a countable set of vertical (horizontal) lines only.*

Cor. 2.—*If either of the two rectangles always contains both sides through P, then S is a countable set.*

7. We now turn to the more complicated cases (1) and (2), specified at the end of § 2, in which the point considered has associated with it one rectangle of definite orientation, and therefore also the vertically opposite rectangle. We treat explicitly the case (1), the case (2) following by the interchange of right and left.

THEOREM 2.—*If S is a plane set of points, each point P of which is the left-hand bottom corner of a rectangle, whose sides are horizontal and vertical, containing no point of S in its interior, then S lies on a countable set of monotone descending curves.*

By Lemma 1, we only have to prove the theorem for the case when the rectangle in question is a square with sides of fixed length 2^{-i} (Fig. 7).

By Lemma 2, P is then also the right-hand top corner of a similar square, free of points of S (Fig. 8).

FIG. 7.

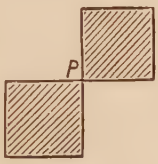

FIG. 8.

As in the preceding proof, we then divide up the plane, chessboard fashion, into chequers, with horizontal and vertical sides of length 2^{-i}, and it is then only necessary to prove the theorem for the sub-set of S in one of these chequers.

The whole chequer is now not covered over, but only the parts of it to the right above and to the left below any point P of the set S in the chequer (that is inside it, or on its left-hand or bottom side, or at its left-hand bottom corner). It immediately follows that, if P and Q are any two points of S in our chequer, P lying to the left of Q, then Q lies below, or on the same level as P. *Hence the line PQ, if not vertical or horizontal, slopes down from P to Q.*

Let T denote the set in our chequer got by closing S. Then, by Lemma 2, T has all the characteristics of S and is in addition closed.

Thus, if on any ordinate there is more than one point of the set T, there will be a highest such point B, and a lowest such point A, and all the points of T to the left of the ordinate lie above or on the same level as B, while all those to the right lie below, or on the same level as A. Thus we may add to T all the points between A and B, without altering the characteristics of the set we are considering. Moreover, if we do this for every ordinate, the set will remain a closed set, since any limiting point of the points introduced must lie between the limiting points of the points A and B on the corresponding ordinates, and is therefore already a point of the enlarged set. Let this enlarged set be

denoted by T'. Then T' is not only a closed set with the given characteristics, but it is also such that on every ordinate in our chequer there is either one point or a closed stretch of points of the set T'.

Let y denote the ordinate of the highest point of T' corresponding to the abscissa x. Then y will be a single-valued function of x, and monotone descending, since, as we saw, the line joining two points (x, y) so determined slopes down from left to right (this was indeed proved for the set S, but follows now, since the characteristic property has never been altered in passing from S to T'). Hence y is continuous with respect to x, except at a countable set of points $x_1, x_2, \ldots$, which are accordingly the values of x for which the sub-set of T' consists of a vertical stretch.

This shows that T' is a monotone descending curve, consisting of the graph of the monotone function y, and the vertical stretches on

$$x = x_1, x_2, \ldots,$$

completing the connection.

Since all the points of our chequer lie on this monotone descending curve, this proves the theorem.

Cor. 1.—*If the given rectangle includes the horizontal (vertical) side through P (so that on this side also there is no other point of the set S), then the monotone curves in question have no horizontal (vertical) parts, and the set S is therefore such that on each horizontal (vertical) line there lies only a countable sub-set of its points.*

8. By projection we can change the vertical lines of our plane into lines inclined at any required acute angle. The result which we obtained in the preceding article being unaltered by this projection, we have at once the following theorem:—

If S is a plane set of points, each point P of which is the lower left-hand corner of a parallelogram, whose sides are drawn in fixed directions, inclined at P at any acute angle, this parallelogram containing no point of S in its interior, then S lies on a countable aggregate of monotone curves.

9. This last theorem enables us to complete our solution of the problem originally proposed. As we divided up the neighbourhood of any point P into four quadrants, and numbered them in order clock-

wise, beginning with the top right-hand quadrant, and so defined the first, second, third and fourth neighbourhoods of the point P, so we shall now bisect each of these quadrants, and so obtain 2^3 new neighbourhoods of P, each having the form of a parallelogram, the angle at P being half a right angle. Beginning with that one of these which lies first to the right of the upward vertical, we number these in order clockwise, the fifth, sixth, ..., twelfth neighbourhoods of the point P (Fig. 9).

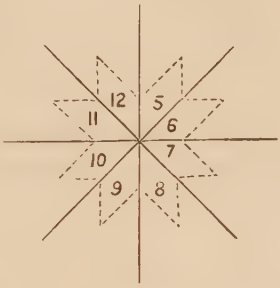

FIG. 9.

Continuing this method of obtaining neighbourhoods, we bisect the angles afresh, and so on. Thus corresponding to each integer i, we have a definite neighbourhood of the point P, of the form of a parallelogram, with an angle at P which decreases down to zero as limit when i approaches infinity.

If now S is a plane set of points, and P, one of its points, is such that there is a triangle with P as corner, containing no point of S in its interior, we shall say that S has *a triangular void at P*. Each such point P will then determine, without ambiguity, the first integer i, such that the i-th neighbourhood of P is empty of points of the set S. In this way the points of S at which the set has a triangular void are distributed into a countably infinite aggregate of sub-sets, each characterised by an integer i, the same for all the points of the sub-set. Each of these sub-sets, being of the type contemplated in the preceding theorem, has the characteristics there specified. We thus have the following theorem :—

THEOREM.—*The sub-set S' of a plane set S at the points of which S has a triangular void lies on a countable aggregate of monotone curves.*

Thus a plane set may be divided up into a sub-set which lies on a countable aggregate of non-oscillatory curves, and a sub-set having the property that, inside every triangle with a point of this sub-set as corner, there are points of the set.

10. If S be a plane set and C its complementary set, we may accordingly divide up the points of the plane into four classes :—

(1) *Internal points of the set S.*

P is said to be an internal point of the set S if all the points of some triangle containing P as internal point are points of S.

(2) *Internal points of the complementary set C.*

(3) *Boundary points of both the sets S and C, at each of which either S or C has a triangular void*; these lie on a countable aggregate of monotone curves.

(4) *Stickle-points of both S and C*; these are boundary points of both sets having the property that in every triangle with one of these points as vertex there are points both of S and of C.

In particular, if S is a region, its boundary points fall into two classes, corresponding to (3) and (4) above; the former of these may be said to constitute the normal boundary of the region. An example of a case when both S and C are regions, and the stickle-points form a set of positive content, is given in our Theory of Sets of Points, p. 246. Another example is that given by Osgood, quoted on p. 183 of the same book, when the perfect set dense nowhere used in the construction has positive content. On the other hand the region called the Chow, given on pp. 184, 185, is one whose boundary consists entirely of the normal boundary, and part of the normal boundary of the complementary region, while this complementary region has no boundary points except those of its normal boundary.

11. So far we have considered the problem entirely from a descriptive point of view. We now turn to metrical considerations.

A monotone curve is known to have plane content zero, therefore also any set which lies on a countable aggregate of monotone curves has content zero. In particular *the sets we have so far determined are all sets of content zero. This leads us to discuss the form of the sub-set of a plane set S consisting of the points of S which are corners of triangles containing only a sub-set of S of zero content.* It is easy to prove that *this sub-set has content zero.*

Consider, in fact, the set of all the triangles containing in their interiors only sub-sets of S of content zero. A countably infinite set of these triangles suffice to cover over all the internal points of the triangles. Let us remove from S all the points internal to these triangles: we have removed a countable aggregate of sets of content zero, that is, a

set of content zero. Let S' be the remaining sub-set of S. Then each of the points P of S, which was the corner of a triangle containing in its interior only a sub-set of S of content zero, is either one of the points already removed, or is such that the triangle in question is now empty of points of S, and therefore of S'. Thus S' has a triangular void at P. It follows by the preceding theorem that these points of S' lie on a countable aggregate of monotone curves, and therefore also form a set of content zero. Removing these also, the sub-set of S removed is of content zero, and the remaining sub-set S'' of S is such that, in every triangle with one of its points as corner, there is a sub-set of S of positive content, and therefore also a sub-set of S'' of positive content. Thus S'' may be said to be homogeneously of positive content.

SMALL CONTRIBUTION TO COMBINATORY ANALYSIS

By Major P. A. MACMAHON.

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THE enumerating generating function for the number of partitions of the number n into m or fewer parts is

$$\frac{1}{(1-x)(1-x^2)(1-x^3)\dots(1-x^m)}.$$

In the Theory of Distributions we consider the number of ways of distributing objects of given specification into boxes of given specification. If there be n objects, their specification may be given by any partition of n. Thus, if the specification be the partition

$$(n_1 n_2 n_3 \dots n_r)$$

of the number n, we mean that n_1 objects are of one kind, n_2 of a second, n_3 of a third, and so on. The objects might be denoted by the factors of the product

$$a_1^{n_1} a_2^{n_2} a_3^{n_3} \dots a_r^{n_r}.$$

Similarly the m boxes might have a specification

$$(m_1 m_2 m_3 \dots m_s),$$

any partition of m. The boxes might be denoted by the factors of the product

$$A_1^{m_1} A_2^{m_2} A_3^{m_3} \dots A_s^{m_s}.$$

The problem of enumerating the partitions of n into m or fewer parts is the same as that of enumerating the distribution of n objects of specification (n) into m or fewer boxes of specification (m).

From this point of view we may generalize the enumeration of ordinary partitions by considering the number of ways of distributing n objects of specification (n) into m or fewer boxes of specification

$$(m_1 m_2 m_3 \dots m_s).$$

We may, without loss of generality, take $m_1, m_2, m_3, \ldots, m_s$ to be integers in descending order of magnitude.

For brevity denote the algebraic fraction

$$\frac{1}{(1-x)(1-x^2)(1-x^3)\ldots(1-x^m}$$

by $$X_m.$$

The slightest consideration suffices to establish that the distributions before us are given by the coefficient of x^n in the expansion of the fraction

$$X_{m_1}X_{m_2}X_{m_3}\ldots X_{m_s},$$

which, if $$(p_1 p_2 p_3 \ldots p_t)$$

be the partition conjugate to

$$(m_1 m_2 m_3 \ldots m_s),$$

may be written

$$\frac{1}{(1-x)^{p_1}(1-x^2)^{p_2}(1-x^3)^{p_3}\ldots(1-x^t)^{p_t}},$$

wherein, of course, $p_1, p_2, p_3, \ldots, p_t$ are integers in descending order of magnitude.

Observe that, if $m_1 = m$, we have

$$(p_1, p_2, p_3, \ldots, p_t) = (1, 1, 1, \ldots, 1),$$

and $t = m$, the case of ordinary partition.

Moreover, if $$m_1 = m_2 = m_3 = \ldots = m_s = 1,$$

we reach the case of distribution of objects of specification (n) into boxes of specification (1, 1, 1, ..., 1), wherein there are s units.

Here there are no similarities between the boxes, and corresponding thereto we have the compositions of the number n into s or fewer parts, the zero part not being excluded.

The generating function is $$(1-x)^{-s},$$

because (s) is the partition conjugate to (1^s).

As an example of the general theorem consider the distribution of objects of specification (n) into boxes of specification

$$(221).$$

Since (32) is the partition conjugate to (221), we derive the generating function

$$(1-x)^{-3}(1-x^2)^{-2}.$$

Expanded to a few terms this is

$$1+3x+8x^2+16x^3+30x^4+50x^5+\dots .$$

To verify the coefficient 30 of x^4, suppose the boxes to be denoted by A, A, B, B, C, and suppose m_A to denote m objects placed in a box A. We have the distributions

4_A	3_A1_A	3_B1_C	2_A2_A	$2_A1_A1_B$	$2_C1_A1_B$	$1_A1_A1_B1_B$
4_B	3_A1_B	3_B1_A	2_B2_B	$2_A1_A1_C$	$2_A1_B1_B$	$1_A1_A1_B1_C$
4_C	3_B1_B	3_C1_A	2_A2_B	$2_B1_B1_C$	$2_B1_A1_A$	$1_A1_B1_B1_C$
	3_A1_C	3_C1_B	2_A2_C	$2_A1_B1_C$	$2_C1_A1_A$	
			2_B2_C	$2_B1_B1_A$	$2_C1_B1_B$	
				$2_B1_A1_C$		

The theorem that has been established is as follows:—

"The coefficient of x^n in the ascending expansion of the algebraic fraction

$$\frac{1}{(1-x)^{p_1}(1-x^2)^{p_2}(1-x^3)^{p_3}\dots(1-x^t)^{p_t}},$$

where $p_1, p_2, p_3, \dots, p_t$ are integers in descending order of magnitude, is equal to the number of ways of distributing n objects of specification (n) into boxes of specification

$$(m_1 m_2 m_3 \dots m_s),$$

where $(m_1 m_2 m_3 \dots m_s)$ is the partition conjugate to the partition

$$(p_1 p_2 p_3 \dots p_t)."$$

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